Formal Solutions of Any-Order Mass, Angular-Momentum, and Dipole Perturbations on the Schwarzschild Background Spacetime

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Abstract

Formal solutions of any-order mass, angular-momentum, and dipole perturbations on the Schwarzschild background spacetime are derived in a gauge-invariant manner. Once we accept the proposal in [K. Nakamura, Class. Quantum Grav. **38** (2021), 145010], we can extend the gauge-invariant linear perturbation theory on the Schwarzschild background spacetime including the monopole ($l = 0$) and dipole ($l = 1$) modes to any-order perturbations of the same background spacetime through the arguments in [K. Nakamura, Class. Quantum Grav. **31** (2014), 135013]. As a result of this resolution, we reached to a simple derivation of the above formal solutions of any order.

Keywords: general relativity, Schwarzschild black hole, any-order gauge-invariant perturbation, monopole mode, dipole mode
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1. INTRODUCTION

Higher-order perturbation theories are topical subjects in recent researches on general relativity, and they have very wide applications to cosmology and gravitational-wave physics. In cosmology, the Planck mission revealed the precise map of the fluctuations of Cosmic Microwave Background (CMB) [1], and the CMB observation is now regarded as a precise science. On the other hand, the direct observation of gravitational waves is accomplished in 2015 [2], and we can expect that a future direction of gravitational-wave science is also precise science through the forthcoming data of many gravitational-wave events. In addition, some projects of space gravitational-wave antenna are also progressing [3, 4]. Among them, the Extreme-Mass-Ratio-Inspiral (EMRI), which is a source of gravitational waves from the motion of a stellar mass object around a supermassive black hole, is a promising target of the Laser Interferometer Space Antenna [3]. To describe the gravitational waves from EMRIs, higher-order black hole perturbation theories are required to support the gravitational-wave physics as a precise science.

In black hole perturbation theories, further sophistication is possible even in perturbation theories on the Schwarzschild background spacetime. There are many studies on the perturbations on the Schwarzschild background spacetime [5, 6] from the works by Regge and Wheeler [7] and Zerilli [8]. In perturbation theories of the Schwarzschild spacetime, we may decompose the perturbations on this spacetime using the spherical harmonics Y_{lm} and classify them into odd- and even-modes based on their parity, because the Schwarzschild spacetime has a spherical symmetry. However, monopole ($l = 0$) and dipole ($l = 1$) modes were separately treated, and their "*gauge-invariant*" treatments were unknown.

In this situation, in [9], we proposed a gauge-invariant treatment of these modes and derived the solutions to the linearized Einstein equations for these modes. Since the obtained solutions in [9] is physically reasonable, we may say that our proposal is also reasonable. In addition, owing to our proposal, the formulation of higher-order gauge-invariant perturbation theory discussed in [10, 11, 12, 13] becomes applicable to any-order perturbations on the Schwarzschild background spacetime.

In this article, we carry out this application and derive the formal solutions of mass ($l = 0$ even mode), angular-momentum ($l = 1$ odd mode), and dipole perturbations ($l = 1$ even mode) to any-order perturbations. We also emphasize that the proposal in [9] is not only for the perturbations on the Schwarzschild background spacetime but also a clue to perturbation theories on a generic background spacetime such as cosmological perturbation theories [18].

The organization of this paper is as follows. In Section 2, we briefly review the framework of the general-relativistic higher-order gauge-invariant perturbation theory [10, 11, 12, 13]. In Section 3, we briefly explain the strategy for gauge-invariant treatments of $l = 0,1$ modes in [9] and summarize the $l = 0,1$ mode solutions which was also derived in [9]. In Section 4, we show the extension of the linear solutions for $l = 0,1$ modes to any-order perturbations. Finally, in Section 5, we provide a brief summary of this paper.

Throughout this paper, we use the unit $G = c = 1$, where G is Newton's constant of gravitation, and c is the velocity of light.

2. GENERAL-RELATIVISTIC HIGHER-ORDER GAUGE-INVARIANT PERTURBATION THEORY

General relativity is a theory based on general covariance, and that covariance is the reason that the notion of "gauge" has been introduced into the theory. In particular, in general-relativistic perturbations, *the second-kind gauge* appears in perturbations, as Sachs pointed out [14]. In general-relativistic perturbation theory, we usually treat the one-parameter family of spacetimes $\{(\mathscr{M}_\lambda, Q_\lambda)|\lambda \in [0,1]\}$ to discuss differences between the background spacetime $(\mathscr{M}, Q_0) = (\mathscr{M}_{\lambda=0}, Q_{\lambda=0})$ and the physical spacetime $(\mathscr{M}_{\rm ph}, \bar{Q}) = (\mathscr{M}_{\lambda=1}, Q_{\lambda=1})$. Here, λ is the infinitesimal parameter for perturbations, $\mathscr{M}_\lambda$ is a spacetime manifold for each λ, and Q_λ is the collection of the tensor fields on $\mathscr{M}_\lambda$. Since each $\mathscr{M}_\lambda$ is a different manifold, we have to introduce the point identification map $\mathscr{X}_\lambda : \mathscr{M} \to \mathscr{M}_\lambda$ to compare the tensor field on different manifolds. This point identification is *the gauge choice of the second kind*. Since we have no guiding principle by which to choose identification map $\mathscr{X}_\lambda$ due to the general covariance, we may choose a different point identification $\mathscr{Y}_\lambda$ from $\mathscr{X}_\lambda$. This degree of freedom in the gauge choice is *the gauge degree of freedom of the second kind. The gauge transformation of the second kind* is a change in this identification map. We note that this second-kind gauge is a different notion of the degree of freedom of coordinate choices on a single manifold, which is called *the gauge of the first kind* [15]. We have to emphasize that the "gauge" which is excluded in our gauge-invariant perturbation theory is not the gauge of the first kind but the gauge of the second kind. In this paper, we call the gauge of the second kind as *gauge* if there is no possibility of confusions.

Once we introduce the gauge choice $\mathscr{X}_k : \mathscr{M} \to \mathscr{M}_\lambda$, we can compare the tensor fields on different manifolds $\{\mathscr{M}_\lambda\}$, and *perturbations* of a tensor field Q_λ are represented by the difference $\mathscr{X}_\lambda^* Q_\lambda - Q_0$, where $\mathscr{X}_\lambda^*$ is the pull-back induced by the gauge choice $\mathscr{X}_\lambda$ and Q_0 is the background value of the variable Q_λ. We note that this representation of perturbations completely depends on the gauge choice $\mathscr{X}_\lambda$. If we change the gauge choice from $\mathscr{X}_\lambda$ to $\mathscr{Y}_\lambda$, the pulled-back variable of Q_λ is then represented by $\mathscr{Y}_\lambda Q_\lambda$. These different representations are related to the gauge-transformation rules as

$$\mathscr{Y}_\lambda^* Q_\lambda = \Phi_\lambda^* \mathscr{X}_\lambda^* Q_\lambda, \quad \Phi_\lambda := \mathscr{X}_\lambda^{-1} \circ \mathscr{Y}_\lambda. \tag{1}$$

Φ_λ is a diffeomorphism on the background spacetime $\mathscr{M}$.

In the perturbative approach, we treat the perturbation $\mathscr{X}_\lambda^* Q_\lambda$ through the Taylor series with respect to the infinitesimal parameter λ as

$$\mathscr{X}_\lambda^* Q_\lambda =: \sum_{n=0}^{k} \frac{\lambda^n}{n!} {}^{(n)}_{\mathscr{X}}Q + O(\lambda^{k+1}), \tag{2}$$

where ${}^{(n)}_{\mathscr{X}}Q$ is the representation associated with the gauge choice $\mathscr{X}_\lambda$ of the kth-order perturbation of the variable Q_λ with its background value ${}^{(0)}_{\mathscr{X}}Q = Q_0$. Similarly, we can have the representation of the perturbation of the variable Q_λ under the gauge choice $\mathscr{Y}_\lambda$, which is different from $\mathscr{X}_\lambda$ as mentioned above. Since these different representations are related to the gauge-transformation rule (1), the order-by-order gauge-transformation rule between nth-order perturbations ${}^{(n)}_{\mathscr{X}}Q$ and ${}^{(n)}_{\mathscr{Y}}Q$ is given from the Taylor expansion of the gauge-transformation rule (1).

Since Φ_λ is constructed by the product of diffeomorphisms, Φ_λ is not given by an exponential map [10, 16, 17], in general. For this reason, Sonego and Bruni [17] introduced the notion of a *knight diffeomorphism* through the following proposition.

Proposition 2.1. *Let Φ_λ be a one-parameter family of diffeomorphisms, and T a tensor field such that $\Phi_\lambda^* T$ is of class C^k. Then, $\Phi_\lambda^* T$ can be expanded around $\lambda = 0$ as*

$$\Phi_\lambda^* T = \sum_{n=0}^{k} \lambda^n \sum_{\{j_i\}\in J_n} C_{n,\{j_i\}} £_{\xi_{(1)}}^{j_1} \cdots £_{\xi_{(n)}}^{j_n} T + O(\lambda^{k+1}). \tag{3}$$

Here, $J_n := \{\{j_i\}|^{\forall} i \in \mathbb{N}, j_i \in \mathbb{N}$, s.t. $\sum_{i=1}^{\infty} i j_i = n\}$ defines the set of indices over which one has to sum in order to obtain the nth-order term, $C_{n,\{j_i\}} := \prod_{i=1}^{n} \frac{1}{(i!)^{j_i} j_i!}$, and $O(\lambda^{k+1})$ is a remainder with $O(\lambda^{k+1})/\lambda^k \to 0$ in the limit $\lambda \to 0$.

The vector fields $\xi_{(1)}, ..., \xi_{(k)}$ in equation (3) are called the generators of Φ_λ. The Taylor expansion (3) is a sufficient representation at least when we concentrate on perturbation theories [13, 17]. Actually, this knight diffeomorphism is suitable for our order-by-order arguments on the gauge issues of general-relativistic higher-order perturbations.

Through the above notion of the knight diffeomorphism, Sonego and Bruni also derived the gauge-transformation rules for nth-order perturbations. As mentioned above, the gauge-transformation rule between the pulled-back variables $\mathscr{Y}_\lambda^* Q_\lambda$ and $\mathscr{X}_\lambda^* Q_\lambda$ is given by equation (1). In perturbation theories, we always use the Taylor expansion of these variables as in equation (2). To derive the order-by-order gauge-transformation rule for the nth-order perturbation, we have to know the form of the Taylor expansion of the pull-back Φ_λ^* of diffeomorphism. Then, we use the general expression (3) of the Taylor expansion of diffeomorphisms. Substituting equations (2) and (3) into equation (1), we obtain the order-by-order expression of the gauge-transformation rules between the perturbative variables ${}^{(n)}_{\mathscr{X}}Q$ and ${}^{(n)}_{\mathscr{Y}}Q$ as

$${}^{(n)}_{\mathscr{Y}}Q - {}^{(n)}_{\mathscr{X}}Q = \sum_{l=1}^{n} \frac{n!}{(n-l)!} \sum_{\{j_i\}\in J_l} C_{l,\{J_i\}} £_{\xi_{(1)}}^{j_1} \cdots £_{\xi_{(l)}}^{j_l} {}^{(n-l)}_{\mathscr{X}}Q. \tag{4}$$

Inspecting the gauge-transformation rule (4), we defined gauge-invariant variables for metric perturbations and for perturbations of an arbitrary tensor field [10, 11]. Since the definitions of gauge-invariant variables for perturbations of an arbitrary tensor field are trivial if we accomplish the separation of the metric perturbations into their gauge-invariant and gauge-variant parts, we may concentrate on the metric perturbations, at first.

We consider the metric $\bar{g}_{ab}$ on the physical spacetime $(\mathscr{M}_{\rm ph}, \bar{Q}) = (\mathscr{M}_{\lambda=1}, Q_{\lambda=1})$, and we expand the pulled-back metric $\mathscr{X}_\lambda^* \bar{g}_{ab}$ to the background spacetime $\mathscr{M}$ through a gauge choice $\mathscr{X}_k$ as

$$\mathscr{X}_\lambda \bar{g}_{ab} = \sum_{n=0}^{k} \frac{\lambda^n}{n!} {}^{(n)}_{\mathscr{X}}g_{ab} + O(\lambda^{k+1}), \tag{5}$$

where $g_{ab} := {}^{(0)}_{\mathscr{X}}g_{ab}$ is the metric on the background spacetime $\mathscr{M}$. The expansion (5) of the metric depends entirely on the gauge choice $\mathscr{X}_\lambda$. Nevertheless, henceforth, we do not explicitly express the index of the gauge choice $\mathscr{X}_\lambda$ if there is no possibility of confusion. In [10, 11], we proposed a procedure to construct gauge-invariant variables for higher-order perturbations. Our starting point to construct gauge-invariant variables was the following conjecture for the linear metric perturbation $h_{ab} := {}^{(1)}g_{ab}$.

Conjecture 2.1. *If the gauge-transformation rule for a tensor field h_{ab} is given by ${}_{\mathscr{Y}}h_{ab} - {}_{\mathscr{X}}h_{ab} = £_{\xi_{(1)}} g_{ab}$ with the background metric g_{ab}, there then exist a tensor field $\mathscr{F}_{ab}$ and a vector field Y^a such that h_{ab} is decomposed as $h_{ab} =: \mathscr{F}_{ab} + £_Y g_{ab}$, where $\mathscr{F}_{ab}$ and Y^a are transformed into ${}_{\mathscr{Y}}\mathscr{F}_{ab} - {}_{\mathscr{X}}\mathscr{F}_{ab} = 0$ and ${}_{\mathscr{Y}}Y^a - {}_{\mathscr{X}}Y^a = \xi_{(1)}^a$ under the gauge transformation, respectively.*

We call $\mathscr{F}_{ab}$ and Y^a the *gauge-invariant* and *gauge-variant* parts of h_{ab}, respectively.

Based on Conjecture 2.1, in [13], we found that the nth-order metric perturbation ${}^{(n)}_{\mathscr{X}}g_{ab}$ is decomposed into its gauge-invariant and gauge-variant parts as[1]

$$\begin{aligned} {}^{(n)}g_{ab} &= {}^{(n)}\mathscr{F}_{ab} \\ &\quad - \sum_{l=1}^{n} \frac{n!}{(n-l)!} \sum_{\{j_i\}\in J_l} C_{l,\{j_i\}} £_{-{}^{(1)}Y}^{j_1} \cdots £_{-{}^{(l)}Y}^{j_l} {}^{(n-l)}g_{ab}. \end{aligned} \tag{6}$$

Furthermore, through the gauge-variant variables ${}^{(i)}Y^a$ $(i = 1, ..., n)$, we also found the definition of the gauge-invariant variable ${}^{(n)}\mathscr{Q}$ for the nth-order perturbation ${}^{(n)}Q$ of an arbitrary tensor field Q. This definition of the gauge-invariant variable ${}^{(n)}\mathscr{Q}$ implies that the nth-order perturbation ${}^{(n)}Q$ of any tensor field Q is always decomposed into its gauge-invariant part ${}^{(n)}\mathscr{Q}$ and gauge-variant part as

$${}^{(n)}Q = {}^{(n)}\mathscr{Q} - \sum_{l=1}^{n} \frac{n!}{(n-l)!} \sum_{\{j_i\}\in J_l} C_{l,\{j_i\}} £_{-{}^{(1)}Y}^{j_1} \cdots £_{-{}^{(l)}Y}^{j_l} {}^{(n-l)}Q. \tag{7}$$

[1] Precisely speaking, to reach the decomposition formula (6), we have to confirm Conjecture 4.1 in [13] in addition to Conjecture 2.1.

For example, the perturbative expansions of the Einstein tensor and the energy-momentum tensor, which are pulled back through the gauge choice $\mathscr{X}_\lambda$, are given by

$$\mathscr{X}_\lambda^* \bar{G}_a{}^b = \sum_{n=0}^{k} \frac{\lambda^n}{n!} {}^{(n)}_{\mathscr{X}} G_a{}^b + O(\lambda^{k+1}), \tag{8}$$

$$\mathscr{X}_\lambda^* \bar{T}_a{}^b = \sum_{n=0}^{k} \frac{\lambda^n}{n!} {}^{(n)}_{\mathscr{X}} T_a{}^b + O(\lambda^{k+1}). \tag{9}$$

Then, the nth-order perturbation ${}^{(n)}_{\mathscr{X}} G_a{}^b$ of the Einstein tensor and the nth-order perturbation ${}^{(n)}_{\mathscr{X}} T_a{}^b$ of the energy-momentum tensor are also decomposed as

$$\begin{aligned} {}^{(n)}G_a{}^b = {}& {}^{(n)}\mathscr{G}_a{}^b \\ & - \sum_{l=1}^{n} \frac{n!}{(n-l)!} \sum_{\{j_i\}\in J_l} C_{l,\{j_i\}} \pounds^{j_1}_{{}^{(1)}_{-}Y} \cdots \pounds^{j_l}_{{}^{(l)}_{-}Y} {}^{(n-l)}G_a{}^b, \end{aligned} \tag{10}$$

$$\begin{aligned} {}^{(n)}T_a{}^b = {}& {}^{(n)}\mathscr{T}_a{}^b \\ & - \sum_{l=1}^{n} \frac{n!}{(n-l)!} \sum_{\{j_i\}\in J_l} C_{l,\{j_i\}} \pounds^{j_1}_{{}^{(1)}_{-}Y} \cdots \pounds^{j_l}_{{}^{(l)}_{-}Y} {}^{(n-l)}T_a{}^b. \end{aligned} \tag{11}$$

Through the lower-order Einstein equation ${}^{(k)}_{\mathscr{X}} G_a{}^b = 8\pi {}^{(k)}_{\mathscr{X}} T_a{}^b$ with $k \le n-1$, the nth-order Einstein equation ${}^{(n)}_{\mathscr{X}} G_a{}^b = 8\pi {}^{(n)}_{\mathscr{X}} T_a{}^b$ is automatically given in the gauge-invariant form

$${}^{(n)}\mathscr{G}_a{}^b = 8\pi {}^{(n)}\mathscr{T}_a{}^b. \tag{12}$$

Here, we note that the nth-order perturbation of the Einstein tensor is given in the form

$${}^{(n)}\mathscr{G}_a{}^b = {}^{(1)}\mathscr{G}_a{}^b\left[{}^{(n)}\mathscr{F}\right] + {}^{(\mathrm{NL})}\mathscr{G}_a{}^b\left[\left\{{}^{(i)}\mathscr{F} \mid i<n\right\}\right], \tag{13}$$

where ${}^{(1)}\mathscr{G}_a{}^b$ is the gauge-invariant part of the linear-order perturbation of the Einstein tensor. Explicitly, ${}^{(1)}\mathscr{G}_a{}^b[A]$ for an arbitrary tensor field A_{ab} of the second rank is given by [11, 15]

$${}^{(1)}\mathscr{G}_a{}^b[A] := {}^{(1)}\Sigma_a{}^b[A] - \frac{1}{2}\delta_a{}^b {}^{(1)}\Sigma_c{}^c[A], \tag{14}$$

$${}^{(1)}\Sigma_a{}^b[A] := -2\nabla_{[a} H_{d]}{}^{bd}[A] - A^{cb} R_{ac}, \tag{15}$$

$$H_{ba}{}^c[A] := \nabla_{(a} A_{b)}{}^c - \frac{1}{2}\nabla^c A_{ab}. \tag{16}$$

As derived in [11], when the background Einstein tensor vanishes, we obtain the identity

$$\nabla_a {}^{(1)}\mathscr{G}_b{}^a[A] = 0 \tag{17}$$

for an arbitrary tensor field A_{ab} of the second rank.

Thus, we emphasize that Conjecture 2.1 was the important premise of the above framework of the higher-order perturbation theory.

3. LINEAR PERTURBATIONS ON THE SCHWARZSCHILD BACKGROUND SPACETIME

We use the 2+2 formulation [6] of the perturbations on spherically symmetric background spacetimes. The topological space of spherically symmetric spacetimes is the direct product $\mathscr{M} = \mathscr{M}_1 \times S^2$, and the metric on this spacetime is

$$g_{ab} = y_{ab} + r^2\gamma_{ab}, \tag{18}$$

$$y_{ab} = y_{AB}(dx^A)_a(dx^B)_b, \quad \gamma_{ab} = \gamma_{pq}(dx^p)_a(dx^q)_b, \tag{19}$$

where $x^A = (t,r)$ and $x^p = (\theta,\phi)$. In addition, γ_{pq} is a metric of the unit sphere. In the Schwarzschild spacetime, the metric (18) is given by

$$y_{ab} = -f(dt)_a(dt)_b + f^{-1}(dr)_a(dr)_b, \tag{20}$$

$$f = 1 - \frac{2M}{r}, \tag{21}$$

$$\gamma_{ab} = (d\theta)_a(d\theta)_b + \sin^2\theta(d\phi)_a(d\phi)_b. \tag{22}$$

On this background spacetime $(\mathscr{M}, g_{ab})$, we consider the components of the metric perturbation as

$$\begin{aligned} h_{ab} = {}& h_{AB}(dx^A)_a(dx^B)_b + 2h_{Ap}(dx^A)_{(a}(dx^p)_{b)} \\ & + h_{pq}(dx^p)_a(dx^q)_b. \end{aligned} \tag{23}$$

In [9], we proposed the decomposition of these components as

$$h_{AB} = \sum_{l,m} \tilde{h}_{AB} S_\delta, \tag{24}$$

$$h_{Ap} = r\sum_{l,m}\left[\tilde{h}_{(e1)A}\hat{D}_p S_\delta + \tilde{h}_{(o1)A}\varepsilon_{pq}\hat{D}^q S_\delta\right], \tag{25}$$

$$\begin{aligned} h_{pq} = r^2\sum_{l,m}\Bigg[& \frac{1}{2}\gamma_{pq}\tilde{h}_{(e0)}S_\delta + \tilde{h}_{(e2)}\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{\Delta}\right)S_\delta \\ & + 2\tilde{h}_{(o2)}\varepsilon_{r(p}\hat{D}_{q)}\hat{D}^r S_\delta\Bigg], \end{aligned} \tag{26}$$

where $\hat{D}_p$ is the covariant derivative associated with the metric γ_{pq} on S^2, $\hat{D}^p := \gamma^{pq}\hat{D}_q$, and $\varepsilon_{pq} = \varepsilon_{[pq]}$ is the totally antisymmetric tensor on S^2.

Note that the decomposition formulae (24)–(26) implicitly state that the Green functions of the derivative operators $\hat{\Delta} := \hat{D}^r\hat{D}_r$ and $\hat{\Delta}+2 := \hat{D}^r\hat{D}_r + 2$ should exist if the one-to-one correspondence between $\{h_{Ap}, h_{pq}\}$ and $\{\tilde{h}_{(e1)A}, \tilde{h}_{(o1)A}, \tilde{h}_{(e0)}, \tilde{h}_{(e2)}, \tilde{h}_{(o2)}\}$ is guaranteed. Because the eigenvalue of the derivative operator $\hat{\Delta}$ on S^2 is $-l(l+1)$, the kernels of the operators $\hat{\Delta}$ and $\hat{\Delta}+2$ are $l=0$ and $l=1$ modes, respectively. Thus, the one-to-one correspondence between $\{h_{Ap}, h_{pq}\}$ and $\{\tilde{h}_{(e1)A}, \tilde{h}_{(o1)A}, \tilde{h}_{(e0)}, \tilde{h}_{(e2)}, \tilde{h}_{(o2)}\}$ is lost for $l=0,1$ modes in decomposition formulae (24)–(26) with $S_\delta = Y_{lm}$. To recover this one-to-one correspondence, in [9], we introduced the mode functions $k_{(\hat{\Delta})}$ and $k_{(\hat{\Delta}+2)m}$ instead of Y_{00} and Y_{1m}, respectively, and consider the scalar harmonic function

$$S_\delta = \begin{cases} Y_{lm} & \text{for } l \ge 2, \\ k_{(\hat{\Delta}+2)m} & \text{for } l = 1, \\ k_{(\hat{\Delta})} & \text{for } l = 0. \end{cases} \tag{27}$$

As the explicit functions of $k_{(\hat{\Delta})}$ and $k_{(\hat{\Delta}+2)m}$, we employ

$$k_{(\hat{\Delta})} = 1 + \delta\ln\left(\frac{1-z}{1+z}\right)^{1/2}, \quad \delta \in \mathbb{R}, \tag{28}$$

$$k_{(\hat{\Delta}+2)m=0} = z\left\{1 + \delta\left(\frac{1}{2}\ln\frac{1+z}{1-z} - \frac{1}{z}\right)\right\}, \tag{29}$$

$$\begin{aligned} k_{(\hat{\Delta}+2)m=\pm1} = {}& (1-z^2)^{1/2} \\ & \times\left\{1 + \delta\left(\frac{1}{2}\ln\frac{1+z}{1-z} + \frac{z}{1-z^2}\right)\right\}e^{\pm i\phi}, \end{aligned} \tag{30}$$

where $z = \cos\theta$. This choice guarantees the linear-independence of the set of the harmonic functions

$$\left\{S_\delta, \hat{D}_p S_\delta, \varepsilon_{pq}\hat{D}^q S_\delta, \frac{1}{2}\gamma_{pq}S_\delta, \right. \\ \left. \left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r\right)S_\delta, 2\varepsilon_{r(p}\hat{D}_{q)}\hat{D}^r S_\delta\right\} \tag{31}$$

including $l = 0,1$ modes if $\delta \neq 0$, but it is singular if $\delta \neq 0$. When $\delta = 0$, we have $k_{(\hat{\Delta})} \propto Y_{00}$ and $\hat{k}_{(\hat{\Delta}+2)m} \propto Y_{1m}$.

Using the above harmonics functions S_δ in equation (27), in [9], we proposed the following strategy.

Proposal 3.1. *We decompose the metric perturbations h_{ab} on the background spacetime with the metric (18)–(22), through equations (24)–(26) with the harmonic functions S_δ given by equation (27). Then, equations (24)–(26) become invertible with the inclusion of $l = 0,1$ modes. After deriving the field equations such as linearized Einstein equations using the harmonic function S_δ, we choose $\delta = 0$ when we solve these field equations as the regularity of the solutions.*

Through this strategy, we can construct gauge-invariant variables and evaluate field equations through the mode-by-mode analyses without special treatments for $l = 0,1$ modes.

Once we accept Proposal 3.1, we reach to the following statement [9].

Theorem 3.1. *If the gauge-transformation rule for a tensor field h_{ab} is given by ${}_{\mathscr{Y}}h_{ab} - {}_{\mathscr{X}}h_{ab} = £_{\xi_{(1)}}g_{ab}$, where g_{ab} is the background metric with the spherical symmetry, then, there exist a tensor field $\mathscr{F}_{ab}$ and a vector field Y^a such that h_{ab} is decomposed as $h_{ab} =: \mathscr{F}_{ab} + £_Y g_{ab}$, where $\mathscr{F}_{ab}$ and Y^a are transformed as ${}_{\mathscr{Y}}\mathscr{F}_{ab} - {}_{\mathscr{X}}\mathscr{F}_{ab} = 0$ and ${}_{\mathscr{Y}}Y^a - {}_{\mathscr{X}}Y^a = \xi^a_{(1)}$ under the gauge transformation.*

Owing to Theorem 3.1, the above general arguments in our gauge-invariant perturbation theory are applicable to perturbations on the Schwarzschild background spacetime including $l = 0,1$ mode perturbations. Furthermore, we derived the $l = 0,1$ solution to the linearized Einstein equation in the gauge-invariant manner [9].

As shown in equation (12), the linearized Einstein equation ${}^{(1)}G_a^{\ b} = 8\pi{}^{(1)}T_a^{\ b}$ for the linear metric perturbation $h_{ab} = \mathscr{F}_{ab} + £_Y g_{ab}$ with the vacuum background Einstein equation $G_a^{\ b} = 8\pi T_a^{\ b} = 0$ is given by

$${}^{(1)}\mathscr{G}_a^{\ b}[\mathscr{F}] = 8\pi{}^{(1)}\mathscr{T}_a^{\ b}. \tag{32}$$

Since we consider the vacuum background spacetime $T_{ab} = 0$, the linear-order perturbation of the continuity equation of the linear perturbation of the energy-momentum tensor is given by

$$\nabla^{a\,(1)}\mathscr{T}_a^{\ b} = 0. \tag{33}$$

We decompose the components of the linear perturbation of ${}^{(1)}\mathscr{T}_{ac}$ as

$${}^{(1)}\mathscr{T}_{ac} = \sum_{l,m}\tilde{T}_{AC}S_\delta(dx^A)_a(dx^C)_c \\ + 2r\sum_{l,m}\left\{\tilde{T}_{(e1)A}\hat{D}_p S_\delta + \tilde{T}_{(o1)A}\varepsilon_{pq}\hat{D}^q S_\delta\right\}(dx^A)_{(a}(dx^p)_{c)} \\ + r^2\sum_{l,m}\left\{\tilde{T}_{(e0)}\frac{1}{2}\gamma_{pq}S_\delta + \tilde{T}_{(e2)}\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}_r\hat{D}^r\right)S_\delta \right. \\ \left. + \tilde{T}_{(o2)}\varepsilon_{s(p}\hat{D}_{q)}\hat{D}^s S_\delta\right\}(dx^p)_a(dx^q)_c. \tag{34}$$

We also derive the continuity equations (33) in terms of these mode coefficients and use these equations when we solve the linearized Einstein equation.

Furthermore, we derived the solutions to the Einstein equation for $l = 0,1$ mode imposing the regularity of the harmonics S_δ through $\delta = 0$. For this reason, we may choose $\tilde{T}_{(e2)} = \tilde{T}_{(o2)} = 0$ for $l = 0,1$ modes. In addition, we may also choose $\tilde{T}_{(e1)A} = 0$ and $\tilde{T}_{(o1)A} = 0$ for $l = 0$ modes due to the same reason. This choice and a component of equation (33) lead to $\tilde{T}_{(e0)} = 0$ for $l = 0$ mode.

Through the above premise, in [9], we derived the $l = 0,1$-mode solutions to the linearized Einstein equations as follows.

For $l = 1$ $m = 0$ odd-mode perturbations, we derived

$$2{}^{(1)}\mathscr{F}_{Ap}(dx^A)_{(a}(dx^p)_{b)} \\ = \left(6Mr^2\int dr\frac{1}{r^4}a_1(t,r)\right)\sin^2\theta(dt)_{(a}(d\phi)_{b)} + £_{V_{(1,o1)}}g_{ab}, \tag{35}$$

where the generator $V^a_{(1,o1)}$ of the term $£_{V_{(1,o1)}}g_{ab}$ in equation (35) is

$$V_{(1,o1)a} = \left(\beta_1(t) + W_{(1,o)}(t,r)\right)r^2\sin^2\theta(d\phi)_a. \tag{36}$$

Here, $\beta_1(t)$ is an arbitrary function of t. The function $a_1(t,r)$ is given by the solutions to the linear-order Einstein equation (32) as follows:

$$a_1(t,r) = -\frac{16\pi}{3M}r^3 f\int dt\tilde{T}_{(o1)r} + a_{10} \\ = -\frac{16\pi}{3M}\int dr r^3\frac{1}{f}\tilde{T}_{(o1)t} + a_{10}, \tag{37}$$

where a_{10} is the constant of integration which corresponds to the Kerr parameter perturbation. Furthermore, $rf\partial_r W_{(1,o)}$ of the variable $W_{(1,o)}$ in equation (36) is determined by the evolution equation

$$\partial_t^2\left(rf\partial_r W_{(1,o)}\right) - f\partial_r\left(f\partial_r\left(rf\partial_r W_{(1,o)}\right)\right. \\ + \frac{1}{r^2}f[3f-1]\left(rf\partial_r W_{(1,o)}\right) = 16\pi f^2\tilde{T}_{(o1)r}. \tag{38}$$

For the $l = 0$ even-mode perturbation, we should have

$${}^{(1)}\mathscr{F}_{ab} = \frac{2}{r}\left(M_1 + 4\pi\int dr\left[\frac{r^2}{f}\tilde{T}_{tt}\right]\right) \\ \times\left((dt)_a(dt)_b + \frac{1}{f^2}(dr)_a(dr)_b\right) \\ + 2\left[4\pi r\int dt\left(\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr}\right)\right](dt)_{(a}(dr)_{b)} \\ + £_{V_{(1,e0)}}g_{ab}, \tag{39}$$

where M_1 is the linear-order Schwarzschild mass parameter perturbation, and $\gamma_1(r)$ is an arbitrary function of r. Here, the generator $V_{(1,e0)a}$ of the term $£_{V_{(1,e0)}}g_{ab}$ in equation (39) is given by

$$V_{(1,e0)a} := \left(\frac{1}{4}f\Upsilon_1 + \frac{1}{4}rf\partial_r\Upsilon_1 + \gamma_1(r)\right)(dt)_a \\ + \frac{1}{4f}r\partial_t\Upsilon_1(dr)_a. \tag{40}$$

In the generator (40), ${}^{(1)}\tilde{F} := \partial_t\Upsilon_1$ satisfies the following equation:

$$-\frac{1}{f}\partial_t^2\tilde{F} + \partial_r(f\partial_r\tilde{F}) + \frac{1}{r^2}3(1-f)\tilde{F} \\ = -\frac{8}{r^3}m_1(t,r) + 16\pi\left[-\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr}\right], \tag{41}$$

where

$$\begin{aligned} m_1(t,r) &= 4\pi \int dr \left[\frac{r^2}{f}\tilde{T}_{tt}\right] + M_1 \\ &= 4\pi \int dt \left[r^2 f \tilde{T}_{rt}\right] + M_1, \quad M_1 \in \mathbb{R}. \end{aligned} \tag{42}$$

For the $l=1$ $m=0$ even-mode perturbation, we should have

$$\begin{aligned} {}^{(1)}\mathscr{F}_{ab} = &-\frac{16\pi r^2 f^2}{3(1-f)}\left[\frac{1+f}{2}\tilde{T}_{rr} + rf\partial_r\tilde{T}_{rr} - \tilde{T}_{(e0)} - 4\tilde{T}_{(e1)r}\right] \\ &\times \cos\theta (dt)_a(dt)_b \\ &+ 16\pi r^2 \left\{\tilde{T}_{tr} - \frac{2r}{3f(1-f)}\partial_t\tilde{T}_{tt}\right\} \\ &\times \cos\theta (dt)_{(a}(dr)_{b)} \\ &+ \frac{8\pi r^2(1-3f)}{f^2(1-f)}\left[\tilde{T}_{tt} - \frac{2rf}{3(1-3f)}\partial_r\tilde{T}_{tt}\right] \\ &\times \cos\theta (dr)_a(dr)_b \\ &- \frac{16\pi r^4}{3(1-f)}\tilde{T}_{tt}\cos\theta\gamma_{ab} + \pounds_{V_{(1,e1)}}g_{ab}, \end{aligned} \tag{43}$$

$$\begin{aligned} V_{(1,e1)a} := &-r\partial_t\Phi_{(e)}\cos\theta(dt)_a \\ &+ \left(\Phi_{(e)} - r\partial_r\Phi_{(e)}\right)\cos\theta(dr)_a \\ &- r\Phi_{(e)}\sin\theta(d\theta)_a, \end{aligned} \tag{44}$$

where $\Phi_{(e)}$ satisfies the following equation:

$$\begin{aligned} &-\frac{1}{f}\partial_t^2\Phi_{(e)} + \partial_r\left[f\partial_r\Phi_{(e)}\right] - \frac{1-f}{r^2}\Phi_{(e)} = 16\pi\frac{r}{3(1-f)}S_{(\Phi_{(e)})}, \\ &S_{(\Phi_{(e)})} := \frac{3(1-3f)}{4f}\tilde{T}_{tt} - \frac{1}{2}r\partial_r\tilde{T}_{tt} + \frac{1+f}{4}f\tilde{T}_{rr} + \frac{1}{2}f^2 r\partial_r\tilde{T}_{rr} \\ &\qquad - \frac{f}{2}\tilde{T}_{(e0)} - 2f\tilde{T}_{(e1)r}. \end{aligned} \tag{45}$$

4. EXTENSION TO THE HIGHER-ORDER PERTURBATIONS

As reviewed in Section 2, the n-th order perturbation of the Einstein equation is given in the gauge-invariant form. We may write this n-th order Einstein equation (12) as follows:

$$\begin{aligned} {}^{(1)}\mathscr{G}_a{}^b\left[{}^{(n)}\mathscr{F}\right] &= -{}^{(\mathrm{NL})}\mathscr{G}_a{}^b\left[\left\{{}^{(i)}\mathscr{F}_{cd} \mid i<n\right\}\right] + 8\pi{}^{(n)}\mathscr{T}_a{}^b \\ &=: 8\pi{}^{(n)}\mathbb{T}_a{}^b. \end{aligned} \tag{46}$$

Here, the left-hand side in equation (46) is the linear term of ${}^{(n)}\mathscr{F}_{ab}$ and the first term on the right-hand side is the nonlinear term consisting of the lower-order metric perturbation ${}^{(i)}\mathscr{F}_{ab}$ with $i<n$. The right-hand side $8\pi{}^{(n)}\mathbb{T}_a{}^b$ of equation (46) is regarded as an effective energy-momentum tensor for the n-th order metric perturbation ${}^{(n)}\mathscr{F}_{ab}$.

The vacuum background condition $G_a{}^b = 0$ implies the mathematical identity (17), and equation (46) implies

$$\nabla^{a}{}^{(n)}\mathbb{T}_a{}^b = 0. \tag{47}$$

This equation gives consistency relations which should be confirmed in concrete physical situations. The first term on the right-hand side in equation (46) does not contain ${}^{(n)}\mathscr{F}_{ab}$. The n-th order perturbation ${}^{(n)}\mathscr{T}_a{}^b$ does not contain ${}^{(n)}\mathscr{F}_{ab}$, either, because our background spacetime is the vacuum. Then, ${}^{(n)}\mathbb{T}_a{}^b$ does not include ${}^{(n)}\mathscr{F}_{ab}$. This situation is the same as that we used when we solved the linear-order Einstein equation (32) with the linear perturbation (33) of the continuity equation of the energy-momentum in [9]. Furthermore, we decompose the tensor ${}^{(n)}\mathbb{T}_{ab}$ as follows:

$$\begin{aligned} {}^{(1)}\mathbb{T}_{ab} =: &\sum_{l,m}\tilde{\mathbb{T}}_{AB}S_\delta(dx^A)_a(dx^B)_b \\ &+ 2r\sum_{l,m}\left\{\tilde{\mathbb{T}}_{(e1)A}\hat{D}_pS_\delta + \tilde{\mathbb{T}}_{(o1)A}\varepsilon_{pq}\hat{D}^qS_\delta\right\}(dx^A)_{(a}(dx^p)_{b)} \\ &+ r^2\sum_{l,m}\left\{\tilde{\mathbb{T}}_{(e0)}\frac{1}{2}\gamma_{pq}S_\delta + \tilde{\mathbb{T}}_{(e2)}\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}_r\hat{D}^r\right)S_\delta\right. \\ &\qquad\left. + \tilde{\mathbb{T}}_{(o2)}\varepsilon_{s(p}\hat{D}_{q)}\hat{D}^sS_\delta\right\}(dx^p)_a(dx^q)_b. \end{aligned} \tag{48}$$

Then, the replacements

$$\begin{aligned} &\tilde{T}_{AB} \to \tilde{\mathbb{T}}_{AB}, \quad \tilde{T}_{(e1)A} \to \tilde{\mathbb{T}}_{(e1)A}, \quad \tilde{T}_{(o1)A} \to \tilde{\mathbb{T}}_{(o1)A}, \\ &\tilde{T}_{(e0)} \to \tilde{\mathbb{T}}_{(e0)}, \quad \tilde{T}_{(e2)} \to \tilde{\mathbb{T}}_{(e2)}, \quad \tilde{T}_{(o2)} \to \tilde{\mathbb{T}}_{(o2)} \end{aligned} \tag{49}$$

in the solutions (35)–(45) yield the solutions to equation (46).

Then, following the strategy as Proposal 3.1 and the results derived in [9], the $l=0,1$-mode solutions to equation (46) are summarized as follows.

For $l=1$ $m=0$ odd-mode perturbations, we should have

$$\begin{aligned} &2{}^{(n)}\mathscr{F}_{Ap}(dx^A)_{(a}(dx^p)_{b)} \\ &= \left(6Mr^2\int dr\frac{1}{r^4}a_n(t,r)\right)\sin^2\theta(dt)_{(a}(d\phi)_{b)} + \pounds_{V_{(n,o1)}}g_{ab}, \end{aligned} \tag{50}$$

where the generator $V^a_{(n,o1)}$ of the term $\pounds_{V_{(n,o1)}}g_{ab}$ in equation (50) is

$$V_{(n,o1)a} = \left(\beta_n(t) + W_{(n,o)}(t,r)\right)r^2\sin^2\theta(d\phi)_a. \tag{51}$$

Here, $\beta_n(t)$ is an arbitrary function of t. The function $a_n(t,r)$ is given by the solutions to the nth-order Einstein equation (46) as follows:

$$\begin{aligned} a_n(t,r) &= -\frac{16\pi}{3M}r^3f\int dt{}^{(n)}\tilde{\mathbb{T}}_{(o1)r} + a_{n0} \\ &= -\frac{16\pi}{3M}\int drr^3\frac{1}{f}{}^{(n)}\tilde{\mathbb{T}}_{(o1)t} + a_{n0}, \end{aligned} \tag{52}$$

where a_{n0} is the constant of integration which corresponds to the Kerr parameter perturbation. Furthermore, $rf\partial_rW_{(n,o)}$ of the variable $W_{(n,o)}$ in equation (51) is determined by the evolution equation

$$\begin{aligned} &\partial_t^2\left(rf\partial_rW_{(n,o)}\right) - f\partial_r\left(f\partial_r\left(rf\partial_rW_{(n,o)}\right)\right. \\ &\quad + \frac{1}{r^2}f\left[3f-1\right]\left(rf\partial_rW_{(n,o)}\right) = 16\pi f^{2(n)}\tilde{\mathbb{T}}_{(o1)r}. \end{aligned} \tag{53}$$

For the $l=0$ even-mode perturbation, we should have

$$\begin{aligned} {}^{(n)}\mathscr{F}_{ab} = &\frac{2}{r}\left(M_n + 4\pi\int dr\left[\frac{r^2}{f}{}^{(n)}\tilde{\mathbb{T}}_{tt}\right]\right) \\ &\times\left((dt)_a(dt)_b + \frac{1}{f^2}(dr)_a(dr)_b\right) \\ &+ 2\left[4\pi r\int dt\left(\frac{1}{f}{}^{(n)}\tilde{\mathbb{T}}_{tt} + f^{(n)}\tilde{\mathbb{T}}_{rr}\right)\right](dt)_{(a}(dr)_{b)} \\ &+ \pounds_{V_{(n,e0)}}g_{ab}, \end{aligned} \tag{54}$$

where M_n is the nth-order Schwarzschild mass parameter perturbation, and $\gamma_n(r)$ is an arbitrary function of r. Here, the generator $V_{(n,e0)a}$ of the term $\pounds_{V_{(n,e0)}} g_{ab}$ in equation (54) is given by

$$V_{(n,e0)a} := \left(\frac{1}{4} f \Upsilon_n + \frac{1}{4} r f \partial_r \Upsilon_n + \gamma_n(r)\right)(dt)_a + \frac{1}{4f} r \partial_t \Upsilon_n (dr)_a, \tag{55}$$

In the generator (55), ${}^{(n)}\tilde{F} := \partial_t \Upsilon_n$ satisfies the following equation:

$$\begin{aligned} & -\frac{1}{f}\partial_t^2 {}^{(n)}\tilde{F} + \partial_r\left(f\partial_r {}^{(n)}\tilde{F}\right) + \frac{1}{r^2} 3(1-f) {}^{(n)}\tilde{F} \\ & \quad = -\frac{8}{r^3} m_n(t,r) + 16\pi\left[-\frac{1}{f} {}^{(n)}\tilde{\mathbb{T}}_{tt} + f {}^{(n)}\tilde{\mathbb{T}}_{rr}\right], \end{aligned} \tag{56}$$

where

$$\begin{aligned} m_n(t,r) &= 4\pi \int dr \left[\frac{r^2}{f} {}^{(n)}\tilde{\mathbb{T}}_{tt}\right] + M_n \\ &= 4\pi \int dt \left[r^2 f {}^{(n)}\tilde{\mathbb{T}}_{rt}\right] + M_n, \quad M_n \in \mathbb{R}. \end{aligned} \tag{57}$$

For the $l=1$ $m=0$ even-mode perturbation, we should have

$$\begin{aligned} {}^{(n)}\mathscr{F}_{ab} = & -\frac{16\pi r^2 f^2}{3(1-f)} \\ & \times \left[\frac{1+f}{2} {}^{(n)}\tilde{\mathbb{T}}_{rr} + r f \partial_r {}^{(n)}\tilde{\mathbb{T}}_{rr} - {}^{(n)}\tilde{\mathbb{T}}_{(e0)} - 4 {}^{(n)}\tilde{\mathbb{T}}_{(e1)r}\right] \\ & \times \cos\theta (dt)_a (dt)_b \\ & + 16\pi r^2 \left\{ {}^{(n)}\tilde{\mathbb{T}}_{tr} - \frac{2r}{3f(1-f)} \partial_t {}^{(n)}\tilde{\mathbb{T}}_{tt} \right\} \\ & \times \cos\theta (dt)_{(a} (dr)_{b)} \\ & + \frac{8\pi r^2 (1-3f)}{f^2(1-f)} \left[{}^{(n)}\tilde{\mathbb{T}}_{tt} - \frac{2rf}{3(1-3f)} \partial_r {}^{(n)}\tilde{\mathbb{T}}_{tt} \right] \\ & \times \cos\theta (dr)_a (dr)_b \\ & - \frac{16\pi r^4}{3(1-f)} {}^{(n)}\tilde{\mathbb{T}}_{tt} \cos\theta \gamma_{ab} + \pounds_{V_{(n,e1)}} g_{ab}, \end{aligned} \tag{58}$$

$$\begin{aligned} V_{(n,e1)a} := & -r\partial_t \Phi_{(n,e)} \cos\theta (dt)_a \\ & + \left(\Phi_{(n,e)} - r\partial_r \Phi_{(n,e)}\right) \cos\theta (dr)_a \\ & - r\Phi_{(n,e)} \sin\theta (d\theta)_a. \end{aligned} \tag{59}$$

These are the main assertion of this article.

5. SUMMARY

In summary, we extended the linear-order solution of the mass perturbation ($l=0$ even mode), the angular-momentum perturbation ($l=1$ odd mode), and the dipole perturbation ($l=1$ even mode) to the any-order formal solutions. Our logic starts from the complete proof of Conjecture 2.1 for perturbations on the Schwarzschild background spacetime. The remaining problem in Conjecture 2.1 was in the treatment of $l=0,1$ modes of the perturbations on the Schwarzschild background spacetime. To resolve this problem, in [9], we introduced the harmonic functions S_δ defined by equation (27) instead of the conventional harmonic function Y_{lm} and proposed Proposal 3.1 as a strategy of a gauge-invariant treatment of the $l=0,1$ perturbations on the Schwarzschild background spacetime. Once we accept this proposal, we reach Theorem 3.1 and we can apply our general arguments of higher-order perturbation theory developed in [10, 11, 12, 13] to perturbations on the Schwarzschild background spacetime.

In [9], we derived the $l=0,1$ solutions (35)–(45) to the linearized Einstein equations following Proposal 3.1. The premise and equations for any-order perturbations are the same as those for the linear perturbations. Then, we reached the formal solutions (50)–(59) for the any-order nonlinear perturbation by the replacements (49).

Of course, the solutions derived here are just formal ones and we have to evaluate the nonlinear terms in the effective energy-momentum tensor ${}^{(n)}\mathbb{T}_a{}^b$, i.e., ${}^{(\mathrm{NL})}\mathscr{G}_a{}^b[\{{}^{(i)}\mathscr{F}_{cd} \mid i<n\}]$ and ${}^{(n)}\mathscr{T}_a{}^b$. This evaluation will depend on the situations which we want to clarify. In addition to the perturbations on the Schwarzschild background spacetime, the strategy in Proposal 3.1 is a clue of the generalization of applications of our general framework on the gauge-invariant higher-order perturbations to other physical situations such as higher-order gauge-invariant cosmological perturbations [18]. We leave further evaluations of our formal solutions (50)–(59) in specific physical situations and the applications to the other perturbation theories with different background spacetimes as future works.

CONFLICTS OF INTEREST

The author declares that there are no conflicts of interest regarding the publication of this paper.

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