Using Embedding Theorems to Account for the Extreme Properties of Traversable Wormholes

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Abstract

Embedding theorems, which have a long history in the general theory of relativity, are used in this paper to account for two of the more troubling aspects of Morris-Thorne wormholes: (1) the origin of exotic matter and the amount needed to sustain a wormhole, and (2) the enormous radial tension that is characteristic of wormholes with moderately sized throats. Attributing the latter to exotic matter ignores the fact that exotic matter was introduced for a completely different reason and is usually present in only small quantities.

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1. INTRODUCTION

Embedding theorems have their origin in classical geometry. Both hyperbolic geometry and elliptic non-Euclidean geometry have an intrinsic constant curvature and can be visualized as the surface of a pseudosphere or ordinary sphere, respectively, in Euclidean three-space. The two-dimensional curved surfaces are thereby embedded in a three-dimensional flat space. More generally, according to Campbell's theorem [1], a Riemannian space can be embedded in a higher-dimensional flat space: an n-dimensional Riemannian space is said to be of embedding class m if $m + n$ is the lowest dimension of the flat space in which the given space can be embedded. Here, $m = \frac{1}{2}n(n+1)$. So, a four-dimensional Riemannian space is of class two since it can be embedded in a six-dimensional flat space. Moreover, a line element of class two can be reduced to a line element of class one by a suitable coordinate transformation [2, 3, 4, 5, 6, 7], discussed further in Section 2.

We continue now by recalling that wormholes are handles or tunnels connecting widely separated regions of our Universe or different universes altogether. Morris and Thorne [8] proposed the following static and spherical symmetric line element for a wormhole spacetime:

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \quad (1)$$

where

$$e^{\lambda(r)} = \frac{1}{1 - \frac{b(r)}{r}}. \quad (2)$$

(We are using units in which $c = G = 1$.) Here, $\nu = \nu(r)$ is called the *redshift function*, which must be everywhere finite to prevent the occurrence of an event horizon. The function $b = b(r)$ is called the *shape function* since it determines the spatial shape of the wormhole when viewed, for example, in an embedding diagram [8]. For the shape function, we must have $b(r_0) = r_0$, where the spherical surface $r = r_0$ is called the *throat* of the wormhole. Other requirements are $b'(r_0) < 1$, called the *flare-out condition*, while $b(r) < r$ for $r > r_0$. A final requirement is asymptotic flatness: $\lim_{r\to\infty}\nu(r) = 0$ and $\lim_{r\to\infty}b(r)/r = 0$.

The flare-out condition can only be met by violating the null energy condition (NEC), $T_{\alpha\beta}k^\alpha k^\beta \geq 0$, for all null vectors k^α, where $T_{\alpha\beta}$ is the energy-momentum tensor. The matter that violates the NEC is called "exotic" in [8]. For the outgoing null vector $(1, 1, 0, 0)$, the violation becomes

$$T_{\alpha\beta}k^\alpha k^\beta = \rho + p_r < 0. \quad (3)$$

Here, $T^t{}_t = -\rho$ is the energy density, $T^r{}_r = p_r$ is the radial pressure, and $T^\theta{}_\theta = T^\phi{}_\phi = p_t$ is the lateral (transverse) pressure.

While the need for exotic matter is rather problematical, it is not a conceptual problem, as we know from the Casimir effect [8]. In other words, exotic matter can be made in the laboratory. An open question is whether enough could be produced to sustain a macroscopic traversable wormhole. It is proposed in this paper that exotic matter may be part of the induced-matter theory, discussed hereinafter.

Our second problem, also to be addressed via the embedding theory, is the enormous radial tension at the throat. To that end, we first need to recall that the radial tension $\tau(r)$ is the negative of the radial pressure $p_r(r)$. It is pointed out in [8] that the Einstein field equations can be rearranged to yield $\tau(r)$: reintroducing c and G for now, the radial tension is given by

$$\tau(r) = \frac{b(r)/r - [r - b(r)]\nu'(r)}{8\pi G c^{-4} r^2}. \quad (4)$$

From this condition, it follows that the radial tension at the throat is

$$\tau(r_0) = \frac{1}{8\pi G c^{-4} r_0^2} \approx 5 \times 10^{41}\,\frac{\text{dyn}}{\text{cm}^2}\left(\frac{10\,\text{m}}{r_0}\right)^2. \quad (5)$$

In particular, for $r_0 = 3$ km, $\tau(r)$ has the same magnitude as the pressure at the center of a massive neutron star [8]. Attributing this outcome to exotic matter ignores the fact that exotic matter was introduced for a completely different reason, ensuring a violation of the NEC. For example, it is known that dark matter and phantom dark energy can support traversable wormholes due to the NEC violation but could not explain the large radial tension. In fact, according to equation (5), to avoid a large $\tau(r_0)$, the throat radius $r = r_0$ would have to be extremely large, so that such a wormhole could only exist on a very large scale.

An even more serious problem is the claim made in [9]: the total amount of exotic matter required can be infinitely small. If so, then the exotic matter cannot be responsible for the large radial tension.

The aforementioned wormholes requiring large throat sizes for their existence have remained topics of interest in cosmology, however, as illustrated in [10, 11]. In a similar vein, reference [12] discusses traversable wormholes in spherical stellar systems (see also [13]). Another area of interest in a dark-matter medium is the possible detection of wormholes by means of gravitational lensing. The determination of the deflection angles of black holes and wormholes is discussed in [14, 15].

Before continuing, let us list the Einstein field equations, referring to line element (1):

$$8\pi\rho = e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2}\right] + \frac{1}{r^2}, \tag{6}$$

$$8\pi p_r = e^{-\lambda}\left[\frac{1}{r^2} + \frac{\nu'}{r}\right] - \frac{1}{r^2}, \tag{7}$$

$$8\pi p_t = \frac{1}{2}e^{-\lambda}\left[\frac{1}{2}\left(\nu'\right)^2 + \nu'' - \frac{1}{2}\lambda'\nu' + \frac{1}{r}\left(\nu' - \lambda'\right)\right]. \tag{8}$$

2. THE ROLE OF EMBEDDING

Embedding theorems have a long history in the general theory of relativity. In particular, according to [16, 17], the vacuum field equations in five dimensions yield the Einstein field equations *with matter*, called the induced-matter theory. What we perceive as matter is simply the impingement of the higher-dimensional space onto ours. Moreover, it is noted in the Introduction that a metric of class two can be reduced to a metric of class one and can therefore be embedded in the five-dimensional flat spacetime:

$$ds^2 = -\left(dz^1\right)^2 + \left(dz^2\right)^2 + \left(dz^3\right)^2 + \left(dz^4\right)^2 + \left(dz^5\right)^2, \tag{9}$$

the coordinate transformation is $z^1 = \sqrt{K}e^{\nu/2}\sinh\frac{t}{\sqrt{K}}$, $z^2 = \sqrt{K}e^{\nu/2}\cosh\frac{t}{\sqrt{K}}$, $z^3 = r\sin\theta\cos\phi$, $z^4 = r\sin\theta\sin\phi$, and $z^5 = r\cos\theta$. As a result,

$$ds^2 = -e^{\nu}dt^2 + \left[1 + \frac{1}{4}Ke^{\nu}(\nu')^2\right]dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{10}$$

Metric (10) is equivalent to metric (1) if

$$e^{\lambda} = 1 + \frac{1}{4}Ke^{\nu}(\nu')^2, \tag{11}$$

where $K > 0$ is a free parameter. The result is a metric of embedding class one. Equation (11) can also be obtained from the Karmarkar condition: [18]

$$R_{1414} = \frac{R_{1212}R_{3434} + R_{1224}R_{1334}}{R_{2323}}, \quad R_{2323} \neq 0.$$

In fact, equation (11) is a solution of the differential equation

$$\frac{\nu'\lambda'}{1 - e^{\lambda}} = \nu'\lambda' - 2\nu'' - (\nu')^2,$$

which is readily solved by the separation of variables. So, K is actually an arbitrary constant of integration.

3. THE WORMHOLE SOLUTION

To help make our wormhole solution physically acceptable, we will use the following redshift function proposed by Lake [19]:

$$\nu(r) = n\ln\left(1 + Ar^2\right), \quad n \geq 1, \tag{12}$$

where A is a positive constant. According to [19], this class of monotone increasing functions generates all regular static spherically symmetric perfect-fluid solutions of the Einstein field equations. Equation (12) has proved to be extremely useful in the study of compact stellar objects [2, 3]. While equation (12) can be written $e^{\nu} = (1 + Ar^2)^n$, it can also be generalized to $e^{\nu} = B(1 + Ar^2)^n$, $B > 0$, since the resulting ν is still a monotone increasing function. For our purposes, however, it is sufficient to let $n = 1$. The resulting form is

$$e^{\nu} = B\left(1 + Ar^2\right), \quad A, B > 0. \tag{13}$$

While A is still a free parameter, the constant B can be determined from the junction conditions discussed at the end of the section.

To see how the free parameter K comes into play, let us determine $b(r)$ from equation (11) by inspection:

$$b(r) = r\left[1 - \frac{1}{1 + \frac{1}{4}Ke^{\nu(r)}[\nu'(r)]^2}\right] + \frac{r_0}{1 + \frac{1}{4}Ke^{\nu(r_0)}\left[\nu'(r_0)\right]^2}, \tag{14}$$

so $b(r_0) = r_0$.

To check the flare-out condition, we start with

$$b'(r) = 1 - \frac{1}{1 + \frac{1}{4}Ke^{\nu}(\nu')^2} + r\frac{\frac{1}{4}Ke^{\nu}\nu'\left[(\nu')^2 + 2\nu''\right]}{\left[1 + \frac{1}{4}Ke^{\nu}(\nu')^2\right]^2}. \tag{15}$$

The last term on the right can be rewritten as

$$\frac{r}{\nu'}\frac{\frac{1}{4}Ke^{\nu}(\nu')^2\left[(\nu')^2 + 2\nu''\right]}{\left[1 + \frac{1}{4}Ke^{\nu}(\nu')^2\right]\left[1 + \frac{1}{4}Ke^{\nu}(\nu')^2\right]}.$$

Since the free parameter K can be arbitrarily large, the 1 in the expression $1 + \frac{1}{4}Ke^{\nu}(\nu')^2$ can be neglected, yielding

$$\frac{r}{\nu'}\frac{(\nu')^2 + 2\nu''}{1 + \frac{1}{4}Ke^{\nu}(\nu')^2}.$$

From equation (13), we also obtain

$$\nu' = \frac{2Ar}{1 + Ar^2}, \qquad \nu'' = \frac{2A - 2A^2r^2}{\left(1 + Ar^2\right)^2}. \tag{16}$$

Substituting in equation (15), we have at the throat

$$\begin{aligned} b'(r_0) &= 1 + \frac{\frac{1 - Ar_0^2}{1 + Ar_0^2}}{1 + \frac{1}{4}Ke^{\nu(r_0)}\left[\nu'(r_0)\right]^2} \\ &= 1 + \frac{1 - Ar_0^2}{1 + Ar_0^2 + KBA^2r_0^2}. \end{aligned} \tag{17}$$

Since K can be made as large as we please,

$$b'(r_0) < 1 \quad \text{for } r_0 > \frac{!}{\sqrt{A}}. \tag{18}$$

So, the flare-out condition is satisfied whenever $r_0 > 1/\sqrt{A}$.

Our final topic in this section is asymptotic flatness. Since $\lim_{r\to\infty}\nu'(r) = 0$ from equation (16), it follows from equation (14) that $\lim_{r\to\infty} b(r)/r = 0$. Unfortunately, it is not true that

$\lim_{r\to\infty} \nu(r) = 0$. The wormhole spacetime must therefore be cut off at some $r = a$ and joined to the exterior Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{19}$$

So,

$$e^{\nu(a)} = B\left(1 + Aa^2\right) = 1 - \frac{2M}{a}, \quad B = \frac{1 - \frac{2M}{a}}{1 + Aa^2}. \tag{20}$$

(We will continue to use B in place of its actual value.) The junction at $r = a$ also yields the mass of the wormhole: $M = \frac{1}{2}b(a)$.

4. THE HIGH RADIAL TENSION AND THE AMOUNT OF EXOTIC MATTER

4.1. The High Radial Tension

Accounting for the high radial tension is one of the goals in this paper. As noted in the Introduction, attributing this property to exotic matter ignores the fact that exotic matter was introduced to ensure a violation of the NEC. The reason for the high radial tension must therefore be sought elsewhere.

Using equation (13) in equation (7), we obtain the following expression for the radial pressure:

$$p_r(r) = \frac{A(2 - KBA)}{8\pi\left(1 + Ar^2 + KBA^2r^2\right)}. \tag{21}$$

Recall that we are primarily interested in relatively small throat sizes, since $\tau(r_0)$ is small for large values of r_0. Now, consider

$$\begin{aligned}\lim_{r_0\to 0} \tau(r_0) &= \lim_{r_0\to 0} \left(-p_r(r_0)\right)\\ &= \lim_{r_0\to 0} \frac{A(-2 + KBA)}{8\pi\left(1 + Ar_0^2 + KBA^2r_0^2\right)}\\ &= \frac{A}{8\pi}(-2 + KBA).\end{aligned} \tag{22}$$

So, $\tau(r_0)$ can be made as large as required due to the free parameter K, as long as r_0 is relatively small.

An alternative to the present theory is discussed in [20]. It is shown that the large radial tension can be accounted for via noncommutative geometry, an offshoot of string theory, or by the existence of a small extra spatial dimension.

4.2. The Amount of Exotic Matter

As noted earlier, since the vacuum field equations in five dimensions yield the Einstein field equations *with matter*, this may very well include exotic matter. So, the amount may seem irrelevant. Given its problematical nature, however, it is desirable to keep the amount to a minimum. The actual amount required to sustain a traversable wormhole was first considered by Visser et al. [9] and then extended by Nandi et al. [21]:

$$\Omega = \int_0^{2\pi}\int_0^{\pi}\int_{r_0}^{\infty} (\rho + p_r)\sqrt{-g}\, dr\, d\theta\, d\phi. \tag{23}$$

Since $p_r(r)$ is negative and large in absolute value, $\rho + p_r$ is also negative, as expected, as the flare-out condition has been met. But we still need to show that $\rho + p_r$ can be small in absolute value. From equations (6) and (21),

$$\begin{aligned}8\pi(\rho + p_r) = {} & \frac{1}{1 + \frac{1}{4}Ke^{\nu}(\nu')^2}\frac{1}{r^2}\left[\frac{r}{\nu'}\left[(\nu')^2 + 2\nu''\right] - 1\right]\\ & + \frac{1}{r^2} + \frac{A(2 - KBA)}{1 + Ar^2 + KBA^2r^2},\end{aligned} \tag{24}$$

which is indeed negative for small r_0 by equation (22). To see the full effect of the free parameter K, we simply observe that

$$\lim_{K\to\infty} \frac{A(2 - KBA)}{1 + Ar^2 + KBA^2r^2} = \frac{-BA^2}{BA^2r^2} = -\frac{1}{r^2} \tag{25}$$

by L'Hospital's rule. So,

$$\lim_{K\to\infty} 8\pi(\rho + p_r) = 0. \tag{26}$$

Since $\rho + p_r$ can be made small thanks to the embedding theory, it follows from equation (23) that the total amount of exotic matter can also be small.

5. CONCLUSIONS

An n-dimensional Riemannian space is said to be of embedding class m if $m + n$ is the lowest dimension of the flat space in which the given space can be embedded. This paper deals with wormholes in spacetimes of embedding class one. The goal is to use the embedding theory to account for two of the most troubling aspects of wormhole physics, the need for exotic matter, and the enormous radial tension at the throat of any moderately sized wormhole.

It is proposed in this paper that the existence of exotic matter can be attributed to the higher-dimensional embedding space, thereby becoming part of the induced-matter theory. So, while the amount of exotic matter may therefore seem irrelevant, its problematical nature demands that the total amount be kept to a minimum. This requirement can be met thanks to the free parameter K.

The large radial tension at the throat is usually attributed to the presence of exotic matter. The problem is that exotic matter was introduced for a totally different reason, the need for violating the NEC. Reducing the amount makes the large radial tension even harder to explain. If the amount is infinitely small [9], this explanation breaks down entirely. The embedding theory has proved to be an effective way to account for the high radial tension without relying on exotic matter.

CONFLICTS OF INTEREST

The author declares that there are no conflicts of interest regarding the publication of this paper.

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