A Recipe for Conformal Blocks

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Abstract

We describe a prescription for constructing conformal blocks in conformal field theories in any space-time dimension with arbitrary quantum numbers. Our procedure reduces the calculation of conformal blocks to constructing certain group theoretic structures that depend on the quantum numbers of primary operators. These structures project into irreducible Lorentz representations. Once the Lorentz quantum numbers are accounted for, there are no further calculations left to do. We compute a multivariable generalization of the Exton function. This generalized Exton function, together with the group theoretic structures, can be used to construct conformal blocks for four-point as well as higher-point correlation functions.

Keywords: conformal field theories, conformal blocks
DOI: 10.31526/LHEP.2022.293

1. INTRODUCTION

Exact results in strongly coupled relativistic field theories are scarce. Analytic treatment is sometimes possible due to additional symmetries extending Poincaré invariance. There are well-known solutions exploiting either conformal symmetry or supersymmetry. Examples are conformal theories in two dimensions and the minimal models in which the scaling dimensions are calculable [1]. Other examples are $\mathcal{N}=2$ supersymmetric theories in four dimensions in which the low-energy gauge coupling is calculable [2]. In both of these cases, additional symmetries constrain the possible forms of interactions, and given such constraints, various consistency conditions are sufficient to obtain analytic solutions.

Conformal symmetry in any number of dimensions restricts the form of two- and three-point correlation functions of primary operators leaving only a finite number of unspecified numerical constants. Four-point and higher-point functions depend on conformally invariant combinations of variables, the conformal cross-ratios. A surprising aspect of conformal field theories (CFTs) is that symmetries also constrain correlation functions beyond three points despite the existence of the invariant cross-ratios for four, or more, coordinates. What differentiates conformal symmetry restrictions on three-point correlation functions from the higher ones is that the quantum numbers of the operators are sufficient to determine the form of the correlator. With four, or more, points, one needs to specify not only the quantum numbers of the operators at each point, the "external" operators, but also additional quantum numbers of "exchange" operators that do not appear explicitly in the correlator. The functional forms of four-point, or higher-point, correlators with given external and exchange quantum numbers are known as the conformal blocks.

Conformal blocks are the main inputs for nonperturbative studies of conformal field theories. The bootstrap approach to conformal field theories invented in the seventies [3] relies on very minimal assumptions such as crossing symmetry and unitarity. A correlation function can be expressed in several ways in terms of the conformal blocks. Since different ways of calculating the same quantity must be equivalent, one obtains constraints on the scaling dimensions of the operators and the three-point function coefficients. In some cases, like the minimal models in two dimensions [1], the bootstrap turns out to be powerful enough to completely determine all parameters. For theories in more two dimensions, there has been a lot of progress in the last decade mostly in numerical bootstrap leading to many interesting results; some examples are in [4].

Beyond two dimensions, the conformal blocks are not known in general. There are several approaches to computing the blocks, but the applicability is often restricted to a particular dimension of space-time or to a particular set of quantum numbers of the operators [5, 6, 7]. Here, we present a unified treatment that yields any conformal block completing the approach we outlined in [8].

We utilize two standard techniques. First is the embedding space in which the d-dimensional space with coordinates x^μ is embedded on the light cone of a $(d+2)$-dimensional projective space [9]. We refer to the embedding space coordinates as η^A with $\eta_A\eta^A=0$ and the identification $\eta^A\sim\lambda\eta^A$ for $\lambda>0$. The position space coordinate is $x^\mu=\eta^\mu/(-\eta^{d+1}+\eta^{d+2})$. Conformal symmetry acts linearly on the η^A coordinates. Second, we successively use the operator product expansion (OPE) inside correlation functions to reduce higher-point functions to lower-point ones. This iterative process is possible because the OPE is convergent inside correlators of conformal theories [10].

These two tricks are old and well known. What allowed us to derive completely general results were two further observations. A careful choice of a differential operator on the embedding space made the necessary calculations manageable. All operators are uplifted to the embedding space in the same way, independently of their quantum numbers, using spinor indices only.

In Section 2, we describe how operators are uplifted to the embedding space and we outline the ingredients of the OPE. The main algebraic results are presented in Section 3 leading to a generalization of the Exton G-function [12]. We illustrate our method with examples in Section 4 and summarize in Section 5. We briefly comment on the special form of the scalar conformal block in even dimensions in Appendix A.

2. OPERATORS, DERIVATIVES, AND THE OPE

Primary operators in CFTs are characterized by their scaling dimensions and Lorentz quantum numbers. We are going to

work exclusively in the embedding space, in which the scaling dimension is simply related to the homogeneity degree of an operator

$$\eta^A \frac{\partial}{\partial \eta^A} \mathcal{O}(\eta) = -\tau_{\mathcal{O}} \mathcal{O}(\eta), \tag{1}$$

where $\tau_{\mathcal{O}}$ is the twist of the operator, which is given in terms of its dimension and spin $\tau_{\mathcal{O}} = \Delta_{\mathcal{O}} - S_{\mathcal{O}}$. For now, we suppressed any information about Lorentz representation of $\mathcal{O}$.

Since the embedding space is larger than the original space, operators with Lorentz indices have additional components that need to be removed. We found it most convenient to represent Lorentz quantum numbers as tensor products of spinors. Any representation of $SO(d+2)$ is contained in a tensor product of the spinor representations. We will not specify the signature of space as it will not play a role here. For simplicity of the presentation, let us assume that d is odd, so we do not need to distinguish different spinor representations. This too is a technical detail that is not relevant. With spinor indices explicit, we will write operators as $\mathcal{O}_{a_1 \dots a_n}(\eta)$.

The following transversality condition

$$\eta_A (\Gamma^A)_a{}^{a_i} \mathcal{O}_{a_1 \dots a_i \dots a_n}(\eta) = 0, \tag{2}$$

imposed on every index $i = 1, \dots, n$ removes the unwanted components of $\mathcal{O}_{a_1 \dots a_n}(\eta)$. In the equation above, Γ^A are the usual Dirac matrices in $d+2$ dimensions. We want the operators to transform in irreducible representations, so we assume that $(\mathcal{P}_{\mathbf{N}})_{a_1 \dots a_n}{}^{b_1 \dots b_n} \mathcal{O}_{b_1 \dots b_n} = \mathcal{O}_{a_1 \dots a_n}$. $\mathcal{P}_{\mathbf{N}}$ is a projection operator from the tensor product of n spinors into any irreducible representation, denoted by $\mathbf{N}$, in that product. Labelling representations by their Dynkin indices, the position space operator with $\mathbf{N}_p = (n_1, \dots, n_r)$ is related to the embedding space representation $\mathbf{N} = (0, n_1, \dots, n_r)$. This way of embedding operators is convenient because the number of spinor indices in the position and embedding spaces is exactly the same. The transversality condition in equation (2) changes the representation from $\mathbf{N}$ to $\mathbf{N}_p$.

Schematically, the OPE of two operators can be written as

$$\mathcal{O}_i(\eta_1)\, \mathcal{O}_j(\eta_2) = \sum_k \sum_{a=1}^{N_{ijk}} {}_a c_{ij}{}^k \, {}_a \mathcal{D}_{ij}{}^k (\eta_1, \eta_2)\, \mathcal{O}_k(\eta_2), \tag{3}$$

where the operators on both sides can be in arbitrary Lorentz representations. The sum over a runs over different possibilities for contracting Lorentz indices of a given set of operators. The number of terms in this sum is the number of independent coefficients in the three-point function $\langle \mathcal{O}_i(\eta_1)\, \mathcal{O}_j(\eta_2)\, \mathcal{O}_k(\eta_3) \rangle$. The operator on the right-hand side of the OPE is assumed to be at η_2 although one could make an equivalent choice of η_1 instead. This operator has to be on the null light cone, so a symmetric choice that treats both coordinates on the same footing is not possible.

The derivative operator ${}_a \mathcal{D}_{ij}{}^k$ in equation (3) serves two goals. It soaks up some number of Lorentz vector indices to ensure that the OPE is Lorentz covariant. It is also needed to ensure that both sides of the OPE have the same degree of homogeneity with respect to coordinates η_1 and η_2.

The derivative operator is not unique as the OPE dictates only the number of vector indices and scaling with respect to the coordinates. One constraint is that the derivatives cannot take fields defined on the light cone outside such light cone. Our choice is driven by computational convenience. The basic building blocks for the derivatives are, see [15] for details,

$$\begin{aligned} \mathcal{D}_{12}^A &= (\eta_1 \cdot \eta_2)^{\frac{1}{2}}\, \mathcal{A}_{12}^{AB} \partial_{2B}, \\ \mathcal{D}_{12}^2 &= \mathcal{D}_{12}^A \mathcal{D}_{12A}, \\ \mathcal{A}_{12}^{AB} &= \frac{1}{(\eta_1 \cdot \eta_2)} \left[(\eta_1 \cdot \eta_2)\, g^{AB} - \eta_1^A \eta_2^B - \eta_1^B \eta_2^A \right]. \end{aligned} \tag{4}$$

The transverse metric $\mathcal{A}_{12}^{AB}$ appears in many places in our construction because $\eta_{1A} \mathcal{A}_{12}^{AB} = \eta_{2A} \mathcal{A}_{12}^{AB} = 0$. The operator that has really convenient properties is

$$\mathcal{D}_{12|h}^A = \frac{\eta_2^A \mathcal{D}_{12}^2}{(\eta_1 \cdot \eta_2)^{\frac{1}{2}}} + 2h \mathcal{D}_{12}^A - h(d+2h-2) \frac{\eta_1^A}{(\eta_1 \cdot \eta_2)^{\frac{1}{2}}}, \tag{5}$$

satisfying, for example, $\mathcal{D}_{12|h+1}^A \mathcal{D}_{12|h}^B = \mathcal{D}_{12|h+1}^B \mathcal{D}_{12|h}^A$ and $\mathcal{D}_{12}^{2h} \mathcal{D}_{12|h'}^A = \mathcal{D}_{12|h+h'}^A \mathcal{D}_{12}^{2h}$. For now, the parameter h is arbitrary, but when $\mathcal{D}_{12|h}^A$ appears in the OPE, h will be uniquely determined.

One of the most useful identities for the derivatives is

$$\mathcal{D}_{12|h+n}^{A_n} \cdots \mathcal{D}_{12|h+1}^{A_1} \mathcal{D}_{12}^{2h} = \frac{\mathcal{D}_{12}^{2(h+n)} \eta_2^{A_1} \cdots \eta_2^{A_n}}{(\eta_1 \cdot \eta_2)^{\frac{n}{2}}}, \tag{6}$$

because it allows us to trade the scalar derivative acting on coordinates for derivatives with Lorentz indices. This combination is so useful that we define

$$\mathcal{D}_{12}^{(d,h,n) A_1 \dots A_n} = \mathcal{D}_{12|h+n}^{A_n} \cdots \mathcal{D}_{12|h+1}^{A_1} \mathcal{D}_{12}^{2h}. \tag{7}$$

The exponent $2h$ of the scalar derivative is not necessarily integer as it is related to the scaling dimensions of the fields.

Having defined the derivatives, it is now possible to write the OPE as

$$\begin{aligned} \mathcal{O}_i(\eta_1)\, \mathcal{O}_j(\eta_2) = \left(\mathcal{T}_{12}^{\mathbf{N}_i} \Gamma \right) \left(\mathcal{T}_{21}^{\mathbf{N}_j} \Gamma \right) \cdot \sum_k \sum_{a=1}^{N_{ijk}} \frac{{}_a c_{ij}{}^k \, {}_a t_{ij}^{12k}}{(\eta_1 \cdot \eta_2)^{p_{ijk}}} \\ \cdot \mathcal{D}_{12}^{(d, h_{ijk} - n_a/2, n_a)} \left(\mathcal{T}_{12 \mathbf{N}_k} \Gamma \right) * \mathcal{O}_k(\eta_2), \end{aligned} \tag{8}$$

where ${}_a c_{ij}{}^k$ are arbitrary coefficients, one for each independent structure, while ${}_a t_{ij}^{12k}$ is a tensor that contracts Lorentz indices of different objects in the OPE. The exponents p_{ijk} and h_{ijk} are determined by comparing the homogeneity of the two sides of the OPE.

The last ingredient of our framework is the half-projectors $\mathcal{T}_{12}^{\mathbf{N}_i} \Gamma$ appearing in equation (8). By construction, these objects are transverse to match the transversality, equation (2), of the operators $\mathcal{O}_i(\eta_1)$ and $\mathcal{O}_j(\eta_2)$. $\mathcal{T}_{12}^{\mathbf{N}_i} \Gamma$ also matches the Lorentz representation of operator $\mathcal{O}_i(\eta_1)$. A simple example will illustrate this concept. Suppose we consider an operator with $\mathbf{N} = (0, 1, 0, \dots)$ that is a two-index antisymmetric tensor. In this case, $\mathcal{T}_{12}^{\mathbf{N}} \Gamma \propto \Gamma^{AB} \eta_{1A} (\mathcal{A}_{12})_{BC}$. We termed these objects half-projectors because one gets a projection operator by contracting together two Γ matrices $\mathcal{P}_{\mathbf{N}} \propto \Gamma^{AB} \Gamma_{AB}$, where the spinor indices of the Γ matrices are free. For more complicated representations, $\mathcal{T}_{12}^{\mathbf{N}_i} \Gamma$ can be constructed recursively; see [14]. It will be important shortly, where as far as coordinate dependence is concerned, the half-projectors are simply polynomials in the coordinates η^A and also contain dot products $(\eta_1 \cdot \eta_2)$.

3. APPLYING THE OPE

Using the OPE in an M-point correlation function reduces it to a function with one fewer point. This can only be of practical use if the derivative operator in equation (8) can be evaluated on the most general function of coordinates that appear in an $M-1$ function. We only need to be concerned with dot products since any variable with free Lorentz indices can be absorbed into derivatives by the identity in equation (6). Thus, the most general expression we need is

$$I_{ij}^{(d,h,n;\mathbf{p})A_1\cdots A_n} = \mathcal{D}_{ij}^{(d,h,n)A_1\cdots A_n} \prod_{a\neq i,j} \frac{1}{\left(\eta_j\cdot\eta_a\right)^{p_a}}, \tag{9}$$

where the derivative operator $\mathcal{D}_{ij}$ is identical to $\mathcal{D}_{12}$ defined in the previous section, except we replaced η_1 with η_i and η_2 with η_j as we are dealing with multiple coordinates.

The natural variables for conformal blocks are the invariant cross-ratios. For $M>3$, we single out two coordinates, η_k and η_l. Our basis for the cross-ratios is

$$x_a = \frac{\left(\eta_i\cdot\eta_j\right)\left(\eta_k\cdot\eta_\ell\right)\left(\eta_i\cdot\eta_a\right)}{\left(\eta_i\cdot\eta_k\right)\left(\eta_i\cdot\eta_\ell\right)\left(\eta_j\cdot\eta_a\right)}, \tag{10}$$

$$z_{ab} = \frac{\left(\eta_i\cdot\eta_k\right)\left(\eta_i\cdot\eta_\ell\right)\left(\eta_a\cdot\eta_b\right)}{\left(\eta_k\cdot\eta_\ell\right)\left(\eta_i\cdot\eta_a\right)\left(\eta_i\cdot\eta_b\right)}, \tag{11}$$

where $a,b\neq i,j$. For convenience, we also define a homogeneous derivative

$$\bar{\mathcal{D}}_{ij;k\ell|h}^{A} = \frac{\left(\eta_i\cdot\eta_j\right)^{\frac{1}{2}}\left(\eta_k\cdot\eta_\ell\right)^{\frac{1}{2}}}{\left(\eta_i\cdot\eta_k\right)^{\frac{1}{2}}\left(\eta_i\cdot\eta_\ell\right)^{\frac{1}{2}}}\mathcal{D}_{ij|h}^{A}. \tag{12}$$

In terms of $\bar{\mathcal{D}}$, we define

$$\bar{I}_{ij;k\ell}^{(d,h,n;\mathbf{p})} = \bar{\mathcal{D}}_{ij;k\ell}^{(d,h,n)} \prod_{a\neq i,j} x_a^{p_a}. \tag{13}$$

By definition, $\bar{I}_{ij;k\ell}^{(d,h,n;\mathbf{p})}$ is homogeneous in every coordinate, and it is proportional to $I_{ij}^{(d,h,n;\mathbf{p})}$ in equation (9).

The expression for $\bar{I}_{ij;k\ell}^{(d,h,n;\mathbf{p})}$ is the central result here. It was obtained mostly by recursion. In the following, we use $(\alpha)_\beta = \Gamma(\alpha+\beta)/\Gamma(\alpha)$ to denote the Pochhammer symbol. We also singled out one of the cross-ratios x_m and traded the remaining x_a's for $y_a = 1 - x_m/x_a$ when $a\neq i,j,m$.

$$\begin{aligned}\bar{I}_{ij;k\ell}^{(d,h,n;\mathbf{p})} &= (-2)^h(\bar{p})_h(\bar{p}+1-d/2)_h x_m^{\bar{p}+h} \\ &\times \sum_{\substack{\{q_r\}\geq 0\\ \bar{q}=n}} S_{(\mathbf{q})} x_m^{\bar{q}-q_0-q_i} K_{ij;k\ell;m}^{(d,h;\mathbf{p};\mathbf{q})}\left(x_m;\mathbf{y};\mathbf{z}\right),\end{aligned} \tag{14}$$

$$\begin{aligned}S_{(\mathbf{q})}^{A_1\cdots A_{\bar{q}}} &= g^{(A_1A_2}\cdots g^{A_{2q_0-1}A_{2q_0}}\bar{\eta}_1^{A_{2q_0+1}}\cdots\bar{\eta}_1^{A_{2q_0+q_1}} \\ &\cdots\bar{\eta}_N^{A_{\bar{q}-q_N+1}}\cdots\bar{\eta}_N^{A_{\bar{q}})},\end{aligned} \tag{15}$$

$$\begin{aligned}K_{ij;k\ell;m}^{(d,h;\mathbf{p};\mathbf{q})} &= \frac{(-1)^{\bar{q}-q_0-q_i-q_j}(-2)^{\bar{q}-q_0}\bar{q}!}{\prod_{r\geq 0} q_r!} \\ &\times \frac{(-h-\bar{q})_{\bar{q}-q_0-q_j}\left(p_m\right)_{q_m}\left(\bar{p}+h\right)_{\bar{q}-q_0-q_i}}{(\bar{p})_{\bar{q}-2q_0-q_i-q_j}(\bar{p}+1-d/2)_{-q_0-q_i-q_j}} \\ &\times \prod_{a\neq i,j,m}\left(p_a\right)_{q_a} K_{ij;k\ell;m}^{(d+2\bar{q}-2q_0,h+q_0+q_j;\mathbf{p}+\mathbf{q})},\end{aligned} \tag{16}$$

$$\begin{aligned}K_{ij;k\ell;m}^{(d,h;\mathbf{p})} &= \sum_{n_a,n_{am},n_{ab}\geq 0} \frac{(-h)_{\bar{n}_m+\bar{n}}\left(p_m\right)_{\bar{n}_m}\left(\bar{p}+h\right)_{\bar{n}-\bar{\bar{n}}}}{(\bar{p})_{\bar{n}+\bar{n}_m}(\bar{p}+1-d/2)_{\bar{n}_m+\bar{\bar{n}}}} \\ &\times \prod_{a\neq i,j,m}\frac{\left(p_a\right)_{n_a} y_a^{n_a} x_m^{n_{am}} z_{am}^{n_{am}}}{n_{am}!(n_a-n_{am}-\bar{n}_a)!y_a^{n_{am}}} \\ &\times \prod_{\substack{a,b\neq i,j,m\\ b>a}}\frac{1}{n_{ab}!}\left(\frac{x_m z_{ab}}{y_a y_b}\right)^{n_{ab}}.\end{aligned} \tag{17}$$

Further details can be found in [14]. It is clear that the expression for $\bar{I}_{ij;k\ell}^{(d,h,n;\mathbf{p})}$ is fairly complicated. However, it is the most general function needed to construct any conformal block with M points.

For $M=4$, $I_{ij}^{(d,h,n;\mathbf{p})}$ in equation (9) is directly related to conformal blocks. For $M>4$, $I_{ij}^{(d,h,n;\mathbf{p})}$ can be used to build the blocks recursively [11]. When $M=4$, there are only two independent cross-ratios, which are x_3 and x_4. In that case, $\mathbf{p} = (p_3,p_4)$, $y_4 = 1-x_3/x_4$, and

$$\begin{aligned}K_{12;34;3}^{(d,h;p_3,p_4)}\left(x_3;y_4\right) &= \sum_{n_4,n_{34}\geq 0}\frac{\left(p_3\right)_{n_{34}}\left(p_4\right)_{n_4}}{\left(p_3+p_4\right)_{n_4+n_{34}}\left(n_4-n_{34}\right)!} \\ &\cdot\frac{(-h)_{n_{34}}\left(p_3+p_4+h\right)_{n_4}}{\left(p_3+p_4+1-d/2\right)_{n_{34}}n_{34}!}y_4^{n_4}\left(\frac{x_3}{y_4}\right)^{n_{34}},\end{aligned} \tag{18}$$

which equals the Exton G-function with the following arguments [12]:

$$G\left(p_4,p_3+p_4+h,p_3+p_4+1-d/2,p_3+p_4;x_3,y_4\right). \tag{19}$$

Function G in equation (19) can also be written as a sum of the fourth Appel functions.[1] For $M=4$, $K_{12;34;3}^{(d,h;p_3,p_4)}(x_3;y_4)$ is the same as Exton G, while for $M>4$, $K_{ij;k\ell;m}^{(d,h;\mathbf{p})}$ generalizes the Exton function to M points.

[1] Exton studied a system of partial differential equations related to the fourth Appel function [12]. Depending on the properties of the solutions near singularities of the differential equations, he defined four functions G, H, K, and L. The relationship of the Exton G-function to the conformal blocks was first pointed out by Dolan and Osborn [13].

4. EXAMPLES

We now turn to some simple examples. A general four-point function can be written as

$$\begin{aligned}&\left\langle \mathcal{O}_i(\eta_1)\,\mathcal{O}_j(\eta_2)\,\mathcal{O}_l(\eta_3)\,\mathcal{O}_m(\eta_4)\right\rangle\\&=\frac{\left(\mathcal{T}_{12}^{\mathbf{N}_i}\Gamma\right)\left(\mathcal{T}_{21}^{\mathbf{N}_j}\Gamma\right)\left(\mathcal{T}_{34}^{\mathbf{N}_l}\Gamma\right)\left(\mathcal{T}_{43}^{\mathbf{N}_m}\Gamma\right)}{(\eta_1\cdot\eta_2)^{r_{12}}(\eta_1\cdot\eta_3)^{r_{13}}(\eta_1\cdot\eta_4)^{r_{14}}(\eta_3\cdot\eta_4)^{r_{34}}}\\&\times\sum_k\sum_{a=1}^{N_{ijk}}\sum_{b=1}^{N_{lmk}}{}_ac_{ij}{}^k\,{}_bc_{lmk}\left(\mathcal{G}_{(a,b)}^{ij|k|lm}\right).\end{aligned}\tag{20}$$

To get this result, we applied the OPE twice: once to the pair $\mathcal{O}_i$ and $\mathcal{O}_j$ and another to the remaining pair $\mathcal{O}_l$, $\mathcal{O}_m$. The sums over k and a are the same as in equation (3), as are the coefficients ${}_ac_{ij}{}^k$. The OPE applied to $\mathcal{O}_l$ and $\mathcal{O}_m$ is responsible for the sum over b; however, there is no second sum because the two-point function is unique ensuring that the index k must be the same for the two OPEs. The powers r_{ij} are uniquely determined by comparing homogeneities of the two sides of equation (20) with respect to the coordinates η.

$\mathcal{G}_{(a,b)}^{ij|k|lm}$ in equation (20) is the four-point conformal block. It is homogenous of degree zero under all four variables η. The conformal block depends on the external operators i, j, l, and m and on the exchange operator k that appeared on the right-hand side of the OPE, as we described earlier. Given the set of operators $(ij|k|lm)$, there can still be several different ways of contracting the Lorentz indices, which are denoted by the subscripts a and b. The conformal blocks are functions of the invariant cross-ratios because of homogeneity. A common choice for the cross-ratios with four variables is

$$u=\frac{(\eta_1\cdot\eta_2)\,(\eta_3\cdot\eta_4)}{(\eta_1\cdot\eta_3)\,(\eta_2\cdot\eta_4)},\qquad v=\frac{(\eta_1\cdot\eta_4)\,(\eta_2\cdot\eta_3)}{(\eta_1\cdot\eta_3)\,(\eta_2\cdot\eta_4)}.\tag{21}$$

The cross-ratios introduced in equation (10) are equal to $x_3=\frac{u}{v}$ and $x_4=u$, while $y_4=1-\frac{1}{v}$.

The simplest case is when none of the operators carry any Lorentz indices, that is, when both external and exchange operators are scalars. With scalar operators only, there are no indices to contract and thus only one block, which is

$$\mathcal{G}_{(1,1)}^{ss|s|ss}=x_3^{\Delta_k+h_{ijk}}K_{12;34;3}^{(d,h_{ijk};-h_{lmk},\Delta_k+h_{lmk})}(x_3;y_4),\tag{22}$$

where $h_{ijk}=-\frac{1}{2}(\Delta_i-\Delta_j+\Delta_k)$. This conformal block can be easily expressed in terms of the Exton G-function as in equation (19). We omitted the overall numerical normalization of the conformal block above since it is unimportant here. However, such normalization is necessary to correctly relate the four-point function to the OPE coefficients in equation (20).

Another instructive case is the conformal block with all scalar external operators, but with the exchange operator of spin ℓ that is transforming in the ℓ-index traceless symmetric representation. This is the only representation of the exchange operator that is allowed by group theory with four external scalar fields. It turns out that the blocks can be written in a very compact form, neglecting overall numerical normalization, as

$$\mathcal{G}_{(1,1)}^{ss|\text{spin }\ell|ss}=C_\ell^{(d/2-1)}(X)\Big|_s,\tag{23}$$

where $C_\ell^{(d/2-1)}$ is the ℓ-order Gegenbauer polynomial with index $d/2-1$. The argument of this polynomial $X=\frac{(\alpha_4-\alpha_2)x_4-(\alpha_3-\alpha_2)x_3}{2}$ is a symbolic expression where the parameters $\alpha_{2,3,4}$ are power-counting variables that are replaced according to a substitution indicated by $|_s$ in equation (23).

The compact expression in equation (23) can be unpacked by expanding $C_\ell^{(d/2-1)}(X)$ in power series and then collecting terms with different powers of α_i and powers of the invariant cross-ratios $x_{3,4}$. The substitution s implies

$$\begin{aligned}&\alpha_2^{s_2}\alpha_3^{s_3}\alpha_4^{s_4}x_3^{r_3}x_4^{r_4}\\&\quad\to\rho^{(d,\frac{\ell+s_2-s_3-s_4}{2};-h_{ijm}-\frac{\ell}{2})}\rho^{(d,h_{ijm}-\frac{s_2-s_3-s_4}{2};\Delta_m)}\\&\qquad\times x_3^{\Delta_m+h_{ijm}-\frac{s_2+s_3-s_4}{2}}x_4^{-s_4}K_{12;34;3}^{(d,a;b,c)}(x_3;y_4),\end{aligned}\tag{24}$$

with $\rho^{(d,h;p)}=(-2)^h(p)_h(p+1-d/2)_h$, $a=h_{ijm}-(s_2-s_3-s_4)/2$, $b=-h_{klm}+(r_3-r_4)/2$, and $c=\Delta_m+h_{klm}-(r_3-r_4)/2$. It is straightforward to use the binomial expansion to express the conformal block for the spin ℓ exchange in equation (23) as a linear combination of the $K_{12;34;3}(x_3;y_4)$ functions with different parameters. For $l=0$, all powers are restricted to $s_{2,3,4}=r_{3,4}=0$, and the expression in equation (23) reduces to equation (22). It turns out that analogous expressions for conformal blocks, written as substitutions on the Gegenbauer polynomials, can be also found for other choices of spins of the operators [17].

Expressions for conformal blocks become rapidly complicated when the operators of interest transform in larger Lorentz representations. With larger representations, the group theoretic tensors carry many indices, and there are many terms involved in contracting all those indices. Such contractions involve the metric tensor and the coordinates η_1 through η_4. Each term that combines various Lorenz representations leads to the $\bar{I}_{ij;k\ell}$ function in equation (14) with arguments that depend on the numbers of different coordinates present. Moreover, general conformal blocks carry indices that contract with the indices of the half-projectors in equation (20).

One of the simplest nontrivial representations is the vector representation, and we will consider operators that transform as vectors in position space. The corresponding half-projector in the embedding space is $\mathcal{T}_{12}^v\Gamma\propto\Gamma^{AB}\eta_{1A}(\mathcal{A}_{12})_{BC}$ that carries one un-contracted vector index C. Our second example is the conformal block for the scalar exchange with two external scalars and two external vectors. This will be denoted as $\mathcal{G}_{(1,1)}^{sv|s|sv}$ as with this choice of operators there is, again, only one possible Lorentz contraction. The block must carry two indices that contract with the half-projectors of the two vector operators. Relatively straightforward algebra leads to

$$\mathcal{G}_{(1,1)}^{sv|s|sv}=\bar{I}_{12;34}^{(d,h_{ijk}-3/2,2;-h_{lmk}+1/2,\Delta_m+h_{lmk}+1/2)}{}_{C_jC_m},\tag{25}$$

where, once again, purely numerical normalization was omitted. C_j and C_m are the indices that contract with the half-projectors of the vector operators $\mathcal{O}_j$ and $\mathcal{O}_m$.

The conformal blocks in equations (22) and (25) can be directly compared with derivations using other methods, and the results agree with [5] and [16], respectively. A number of further, significantly more involved, derivations of four-point conformal blocks are presented in [17].

5. DISCUSSION

There are several advantages of the formalism described in this article for computing conformal blocks. Every Lorentz representation appears on the same footing: there is no significant distinction between bosonic and fermionic operators. All operators carry only spinor indices, with an even number of indices for bosons and an odd number for fermions. The particulars of the representation are encoded in the half-projectors $\mathcal{T}_{12}^{\mathbf{N}_i}\Gamma$ introduced in equation (8). The half-projectors are functions of coordinates and the Γ matrices of $SO(d+2)$. The half-projectors are straightforward to write for any fundamental representation. Half-projectors for larger representations can be constructed recursively starting from the smaller ones.

Given the explicit form for $\bar{I}_{ij;k\ell}^{(d,h,n;\mathbf{p})}$ in equation (13), no evaluations are needed to obtain a conformal block. There is no need to solve differential equations or compute integrals. The problem has been reduced to putting together Lego bricks. The half-projectors and $\bar{I}_{ij;k\ell}^{(d,h,n;\mathbf{p})}$ form a complete Lego set for conformal blocks. We are not implying that obtaining blocks with large Lorentz representations, or many points, is trivial, but that our formalism offers a prescription for how to do that and provides all the necessary ingredients.

$\bar{I}_{ij;k\ell}^{(d,h,n;\mathbf{p})}$ is very complicated. Part of that complication stems from its generality. It can be used for M-point functions, not just four-point functions one might be most interested in. There are only two cross-ratios for four points, as discussed in the previous section. Hence, the set of cross-ratios x_m, $\mathbf{y}$, and $\mathbf{z}$, in equation (13), reduces to just two: one x one y, and no z's. We computed the most general function because the methods that yielded the answer for four points were sufficient to extend the answer to M points and the corresponding $M(M-3)/2$ cross-ratios.

The OPE plays the primary role in our approach. It extends $(M-1)$-point functions to M-point functions. Focusing on four-point functions alone, one could use the OPE only once since three-point functions can be written relatively easily. It is, however, possible to construct every possible correlator using the OPE starting with the two-point function. The two-point function follows the OPE of two operators with the identity operator on the right-hand side. Moving on to three points, one can construct a basis for the three-point functions and relate them to the OPE coefficients. The conformal blocks at four, and more, points are the next steps in employing the OPE.

The functions $K_{ij;k\ell;m}^{(d,h;\mathbf{p};\mathbf{q})}$ and its little sibling $K_{ij;k\ell;m}^{(d,h;\mathbf{p})}$ in equation (13) have a number of interesting properties [14]. For example, contiguous relations express $K_{ij;k\ell;m}^{(d,h;\mathbf{p})}$ in terms of $K_{ij;k\ell;m}^{(d+2,h;\mathbf{p}')}$. Such relations suggest associations between conformal blocks that we are only starting to explore.

We hope that our methods will lead to further advances in conformal bootstrap. The numerical bootstrap can benefit from derivations of previously unknown conformal blocks. It might also be possible to formulate analytic bootstrap completely in the embedding space, which seems more natural for CFTs compared to the position space.

Appendix A. EVEN DIMENSIONS

Dolan and Osborn in their seminal papers [5] pointed out that the scalar conformal blocks are simplified in even dimensions when the invariant cross-ratios are expressed in terms of z and $\bar{z}$, which are defined as $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$. In these variables, the blocks can be expressed as sums of terms that factorize into products of the ${}_2F_1$ hypergeometric functions that depend on z and $\bar{z}$.

It might be of interest to know how the result for even dimensions follows from equations (23) and (24). This is, however, not completely straightforward. The direct route of changing variables from x_3 and y_4 to z and $\bar{z}$ involves multiple power series resummations. Although the brute force approach works for a specific numbers of dimensions, it is not at all illuminating and much too long to be presented here.

A better approach is utilizing the recurrence relation for the Gegenbauer polynomials:

$$C_\ell^\alpha(X) = \frac{1}{\ell}\left[2X(\ell+\alpha-1)C_{\ell-1}^\alpha(X) - (\ell+2\alpha-2)C_{\ell-2}^\alpha(X)\right].$$

This leads to a recurrence relation for the blocks, which is described in detail [17]. The recurrence relation in [17] is identical to the relation in [5]. Since the recurrence relates blocks with spin ℓ exchange to those with smaller ℓ's, it is enough to check that the $\ell = 0$ block derived using our method coincides with the block in [5]. The results for even dimensions for $\ell > 0$ in terms of z and $\bar{z}$ then follow from the recurrence relation.

CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this letter.

ACKNOWLEDGMENTS

We thank W. Goldberger, W. J. Ma, and V. Prilepina for helpful discussions. The work of JFF is supported by NSERC and FRQNT.

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