

Quantum Newtonian Cosmology Revisited

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Abstract

We formulate the Lagrangian of the Newtonian cosmology where the cosmological constant is also introduced. Following the affine quantization procedure, the Hamiltonian operator is derived. The wave functions of the Newtonian universe and the corresponding eigenvalues for the case of matter dominated by a negative cosmological constant are given.

Keywords: Newtonian cosmology, affine quantization, cosmological constant

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1. INTRODUCTION

Defining the universe as all of space and time and their contents, including planets, stars, galaxies, and all other forms of matter and energy, some natural questions arise about the beginning, the existence of the universe, its present constituents, and how the life will be in the future. The field of cosmology is the best candidate to respond to those questions as cosmology seeks to answer these questions that are the oldest of mankind. The study of the cosmos is as old as humanity and has always been fascinating. Physical cosmology is the scientific study of the universe as a whole based on the laws of physics, to which Einstein's equations are very likely to give a correct description. General relativity describes the observed universe very well. However, it is a complicated theory, and often, it is difficult to find solutions. The Newtonian theory, on the other hand, is in many ways far simpler and is very often able to satisfactorily model the universe.

The Newtonian cosmology began with the work of McCrea and Milne in 1934 in which simple isotropic and homogeneous universes were studied [1, 2]. The two assumptions fit well with the observable universe, namely, the expanding universe free of rotation and shear. There are two approaches to Newtonian cosmology: the first one deals with potential function worked out by McCrea and Milne [1, 2], Gödel [3], Herkmann and Schücking [4, 5], and Raychaudhury [6]. The second approach uses gravitational force given by Narlikar [7]. A summary of Newton's cosmology is given by Bondi [8], North [9], and Ellis [10].

The application of quantum theory to the description of the universe as a whole is called quantum cosmology. Quantum Newtonian cosmology was proposed by Freedman et al. [11]. It is possible to construct a wave function for Newtonian cosmology in the framework of nonrelativistic quantum mechanics [12, 13, 14, 15]. We are interested in quantum cosmology and a good way to start is to revisit quantum Newtonian cosmology in the sense of [14]. The motivation in revisiting [14] is that the variable " a ", that is the scale factor is positive and the configuration space for our discussions is the positive half line; the wave function in that case is defined only in the positive axis. This leads one to require appropriate boundary conditions or

to consider the appropriate quantization procedure that is the affine quantization.

Our work is presented as follows. In Section 2, we present the classical model, where we start by recalling the cosmological principle and then formulate the Lagrangian like in [16]. In Section 3, we perform the affine quantization procedures and formulate the quantum Hamiltonian operator. In Section 4, the wave functions of the Newtonian universe and the energy levels are given. Section 5 is about some concluding remarks. An appendix about the Biconfluent Heun equation and its polynomial form of solution is added after the concluding remarks.

2. THE CLASSICAL MODEL

2.1. Cosmological Principle

In modern physics, cosmology begins with the application of Einstein's theory of gravity, or general relativity, to the universe. This is a difficult task and would probably not be possible without a basic assumption about the universe called the cosmological principle that says that "On large (but not too large) scales, the universe is homogeneous and isotropic." The statement of isotropy means that the universe is the same in all directions (the universe looks the same whether you are looking directly outward from the North Pole or the South Pole). The homogeneity means that the universe is the same at all points. The statements of homogeneity and isotropy are distinct but closely related: for example, a universe that is isotropic around multiple points will be homogeneous while a universe that is homogeneous may not be isotropic. A universe that is only isotropic around one point is not homogeneous. Without the cosmological principle, much of our presumed understanding of the workings of the universe would be invalid.

2.2. Lagrangian Formulation

We consider a cosmological model based on Newtonian dynamics. It uses an approach worked out by Milne and McCrea in which it has no sense to associate the gravitational phenomena with the effects of spacetime curvature [1, 2]. In this context, it was shown that the universe's behavior could be understood on the basis of classical physics, which does not use the mathematical complexity in the study of the universe as in the Einsteinian cosmology. The simplest model of the universe one can think of is that of a universe filled with dust (nonrelativistic pressureless matter, $p = 0$). We can think of this dust as a collection of point particles and on the cosmological scales

of approximation 10–1000 Mpc [17]. These point particles are a sufficiently good approximation for galaxies or even galaxy clusters. Consider the evolution of a small spherical portion of the universe, where Newton's theory applies: then the behavior of this portion will reflect the evolution of the universe.

The cosmological principle states that there is no preferred place or direction in the universe on large scale, so we can pick any coordinate system with respect to which we can measure the positions and velocities of these test particles.

Let us consider a galaxy that we consider as a particle of mass m for our Newtonian system located at a radius $a(t)$ from an arbitrarily defined origin, where $a(t)$ is a distance we refer to as the scale factor. This scale factor acts to simply scale up or scale down our Newtonian universe. We assume that the motion about the origin must be spherically symmetric. The force of gravity on the particle (galaxy) at distance $a(t)$ coming from the mass of the homogeneous universe inside the sphere of radius $a(t)$ is the same as if all the masses were at the center of the sphere. There is no force arising from the region outside the sphere.

In fixed rectangular coordinates (comoving), the kinetic energy for the motion of the particle (galaxy) is given by

$$T = \frac{1}{2}m\dot{a}^2. \quad (1)$$

We assume that the particle (galaxy) moves in a conservative force field, so the potential energy is given by

$$U = -\frac{GMm}{a}. \quad (2)$$

In both Newtonian cosmology and relativistic cosmology, the universe is unstable to gravitational collapse. Both Newton and Einstein believed the universe is static. In order to obtain this, Einstein introduced a repulsive gravitational force:

$$F_\Lambda = \frac{\Lambda}{3}ma, \quad (3)$$

where the constant Λ is called the cosmological constant. When $\Lambda > 0$, this force is pointed radially outward, repulsive relative to the point origin. If $\Lambda < 0$, the force is attractive relative to the point origin. We can also introduce in the Newtonian approach to cosmology a term containing the cosmological constant, associated with a kind of cosmological force. We assume then that there exists a global cosmological force that affects the particle (galaxy). We take this force to be the repulsive gravitational force that yields the potential

$$U_\Lambda = -\int_0^a F_\Lambda da' = -\frac{1}{6}\Lambda ma^2. \quad (4)$$

In fixed rectangular coordinates (comoving), the Lagrangian of the system is

$$L(a, \dot{a}) = \frac{1}{2}m\dot{a}^2 + \frac{GMm}{a} + \frac{1}{6}\Lambda ma^2. \quad (5)$$

The Lagrangian depends only on the scale factor and its derivative, and is independent of the time because the system is under the action of a uniform force field.

3. AFFINE QUANTIZATION AND HAMILTONIAN OPERATOR

The constant quantity of the motion is the classical Hamiltonian of the system defined by

$$H = \frac{\partial L}{\partial \dot{a}} \dot{a} - L. \quad (6)$$

Inserting equation (5) into equation (6), we got the Hamiltonian at the classical level:

$$H(a, p_a) = \frac{1}{2m}p_a^2 - \frac{GMm}{a} - \frac{1}{6}\Lambda ma^2, \quad (7)$$

where p_a is considered as the corresponding momentum:

$$p_a = \frac{\partial L}{\partial \dot{a}} = m\dot{a}. \quad (8)$$

We consider fixed rectangular comoving coordinates in which we can define the phase space as (a, p_a) with p_a in equation (8) being the linear momentum. The associate Hamiltonian in terms of the phase space coordinates is given by equation (7). Since variable a is positive, we consider the positive axis. In that sense, we choose to perform the affine quantization procedure due to J. R. Klauder [18, 19, 20] that received some recent applications [21, 22, 23, 24, 25, 26, 27]. Why is there the choice of the affine quantization procedure instead of the canonical quantization one? Our motivation is due to the fact that there is a difficulty with canonical quantization when it comes to configuration spaces other than \mathbb{R}^n . For example, here our variable a is positive and then we consider the configuration space to be \mathbb{R}^+ . It seems reasonable to use a and p_a as classical observables, which satisfy the usual commutation relations. However, when we try to represent these by operators $\hat{a} \equiv a$ and $\hat{p}_a \equiv -i\hbar \frac{\partial}{\partial a}$, it turns out that the momentum operator \hat{p}_a is not self-adjoint on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^+, da)$. Thus, a straightforward application of Dirac's canonical recipe is impossible. Affine quantization involves a simple recombination of classical variables expanding similar procedures of canonical quantization as it is described here as follows.

The nonvanishing Poisson bracket of the canonical coordinates is given by

$$\{a, p_a\} = 1. \quad (9)$$

Multiplying the equation (9) by a , we have

$$a \{a, p_a\} = a, \quad (10)$$

and setting $d_a = ap_a$, we have

$$\{a, d_a\} = a, \quad (11)$$

$$H(a, d_a) = \frac{1}{2m}d_a a^{-2}d_a - \frac{GMm}{a} - \frac{1}{6}ma^2. \quad (12)$$

The variables a, d_a are not canonical variables but form a Lie algebra and then worthy to be considered as a new pair of classical variables. The canonical quantization involves \hat{a} and \hat{p}_a which are self-adjoint operators that satisfies the commutation relations

$$[\hat{a}, \hat{p}_a] = i\hbar, \quad (13)$$

and from the canonical quantization, it follows that

$$[\hat{a}, \hat{d}_a] = i\hbar\hat{a}, \quad (14)$$

where

$$\hat{d}_a = \frac{1}{2} (\hat{a}\hat{p}_a + \hat{p}_a\hat{a}) \quad (15)$$

is called the dilation operator because it dilates \hat{a} rather than translates \hat{a} as \hat{p}_a does. The operators \hat{a} and \hat{d}_a are realized as follows:

$$\hat{a}\psi(a) = a\psi(a); \quad \hat{d}_a\psi(a) = -i\hbar \left(a \frac{d}{da} + \frac{1}{2} \right) \psi(a), \quad (16)$$

where $\psi(a)$ is the wave function of the Newtonian universe. The Hamiltonian operator for a particle (galaxy) moving in the Newtonian universe is then given by

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{da^2} + \frac{\hbar^2}{2m} \frac{3}{4} \frac{1}{a^2} - \frac{GMm}{a} - \frac{\Lambda}{6} ma^2. \quad (17)$$

By using the affine quantization procedure, we gain an extra term in the Hamiltonian equation (17), that is, $\frac{\hbar^2}{2m} \frac{3}{4} \frac{1}{a^2}$, compare the similar Hamiltonian in [14] where canonical quantization and appropriate boundary conditions have been required. This difference is due to affine quantization and thus some differences in the energy levels and wave functions should be also expected.

4. THE WAVE FUNCTIONS OF THE NEWTONIAN UNIVERSE AND THE ENERGY LEVELS

Once we have the Hamiltonian operator of the system, it is possible to find the wave functions of the Newtonian universe [13, 14, 15], by solving the time-independent Schrödinger equation

$$\hat{H}\psi(a) = E\psi(a), \quad (18)$$

where $\psi(a)$ corresponds to the eigenvalues E , with $\Psi(a, t) = \psi(a)e^{-iEt/\hbar}$ being the general wave function solution of the time-dependent Schrödinger equation.

Equation (18) is equivalent to

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(a)}{da^2} + \left(\frac{\hbar^2}{2m} \frac{3}{4} \frac{1}{a^2} - GMm \frac{1}{a} - \frac{\Lambda}{6} ma^2 \right) \psi(a) = E\psi(a). \quad (19)$$

Let us solve equation (19) that is equivalent by dividing by $\frac{\hbar^2}{2m}$ to

$$-\frac{d^2\psi(a)}{da^2} + \frac{2m}{\hbar^2} \left(\frac{\hbar^2}{2m} \frac{3}{4} \frac{1}{a^2} - GMm \frac{1}{a} - \frac{\Lambda}{6} ma^2 \right) \psi(a) = \frac{2m}{\hbar^2} E\psi(a). \quad (20)$$

Labeling the parameters as follows:

$$\alpha = \frac{3}{4}; \quad \delta = -\frac{4GMm^2}{\hbar^2}; \quad \omega^2 = \left(\frac{-\Lambda}{3} \right) \frac{m^2}{\hbar^2}; \quad k^2 = \frac{2mE}{\hbar^2}, \quad (21)$$

we have

$$-\frac{d^2\psi(a)}{da^2} + \left(\frac{\alpha}{a^2} + \frac{1}{2} \frac{\delta}{a} + \omega^2 a^2 - k^2 \right) \psi(a) = 0. \quad (22)$$

The sign of the cosmological constant is for the moment left arbitrary. In search of a solution of equation (22), we proceed

like the way we solved similar equations in [25] by setting the following Ansatz:

$$\psi(a) = a^{\beta+1} e^{-\frac{\omega}{2} a^2} v(a), \quad (23)$$

where $\beta(\beta+1) = \alpha$ and the function $v(a)$ is an auxiliary function satisfying the equation

$$v''(a) + \left(2(\beta+1) \frac{1}{a} - 2\omega a \right) v'(a) + \left(k^2 - \omega(2\beta+3) - \frac{1}{2} \frac{\delta}{a} \right) v(a) = 0. \quad (24)$$

Performing the change of variable $x = \omega^{1/2} a$ in equation (24), we have

$$v''(x) + \left(\frac{2(\beta+1)}{x} - 2x \right) v'(x) + \left(\frac{k^2}{\omega} - (2\beta+3) - \frac{1}{2} \frac{\delta\omega^{-1/2}}{x} \right) v(x) = 0. \quad (25)$$

Equation (25) is a particular case of the biconfluent Heun equation [28], and its solution is the biconfluent Heun function:

$$v(x) = \text{HeunB} \left(2\beta+1; 0; k^2\omega^{-1}; \delta\omega^{-1/2}; x \right). \quad (26)$$

Some details about biconfluent Heun equation are given in the appendix (see after Section 5). From the relation $\beta(\beta+1) = \alpha$ and $\alpha = 3/4$, there are two possibilities $\beta = 1/2$ or $\beta = -3/4$. For $(2\beta+1)$ not a negative integer, the biconfluent Heun functions $v(x)$ can be written in series form as

$$v(x) = \text{HeunB} \left(2\beta+1; 0; k^2\omega^{-1}; \delta\omega^{-1/2}; x \right) \quad (27)$$

$$= \sum_{p=0}^{\infty} \frac{A_p}{(1+(2\beta+1))_p} \frac{x^p}{p!}, \quad (28)$$

where

$$A_{p+2} = \frac{1}{2} \delta\omega^{-1/2} A_{p+1} - (p+1)(p+1+(2\beta+1)) \times \left(k^2\omega^{-1} - (2\beta+1) - 2 - 2p \right) A_p. \quad (29)$$

We consider the value $\beta = 1/2$ that gives $2\beta+1 = 2$ and then

$$v(x) = \text{HeunB} \left(2; 0; k^2\omega^{-1}; \delta\omega^{-1/2}; x \right) \quad (30)$$

$$= \sum_{p=0}^{\infty} \frac{A_p}{(3)_p} \frac{x^p}{p!}, \quad (31)$$

where

$$A_{p+2} = \frac{1}{2} \delta\omega^{-1/2} A_{p+1} - (p+1)(p+3) \left(k^2\omega^{-1} - 4 - 2p \right) A_p. \quad (32)$$

From the recursion equation (32), the function $v(x)$ becomes a polynomial of degree n and the series terminate if and only if the two following conditions are fulfilled:

$$(1) \quad k^2\omega^{-1} - 4 = 2n,$$

$$(2) \quad A_{n+1} = 0,$$

where A_{n+1} is a polynomial of degree $(n+1)$ in $\tilde{\delta} = \delta\omega^{-1/2}$. There are at most $(n+1)$ suitable values of value $\tilde{\delta}$ usually labeled as $\tilde{\delta}_\mu^n$, $0 \leq \mu \leq n$. In that sense, we have

$$v_n(x) = \text{HeunB}\left(2; 0; 2(n+2); \tilde{\delta}_\mu^n; x\right) \quad (33)$$

$$= P_{n,\mu}(2, 0, x) = \sum_{p=0}^n \frac{A_p}{(3)_p} \frac{x^p}{p!}, \quad (34)$$

with

$$A_0 = 1; \quad A_1 = \frac{1}{2}\tilde{\delta}, \quad (35)$$

$$A_{p+2} = \frac{1}{2}\tilde{\delta}A_{p+1} - 2(p+1)(p+3)(n-p)A_p; \quad A_{n+1} = 0. \quad (36)$$

The eigenvalues are determined from the condition $k^2\omega^{-1} - 4 = 2n$ as follows:

$$E_n = (n+2)\hbar \left(\frac{-\Lambda}{3}\right)^{\frac{1}{2}}. \quad (37)$$

The associated physically acceptable wave functions $\psi_n(a)$ are given by

$$\psi_n(a) = C_n a^{3/2} e^{-\frac{\omega}{2}a^2} \text{HeunB}\left(2; 0; 2(n+2); \tilde{\delta}_\mu^n; \omega^{\frac{1}{2}}a\right). \quad (38)$$

The constant C_n are arbitrary factors that can be determined under the normalization condition

$$\int_0^\infty |\psi_n(a)|^2 da = 1. \quad (39)$$

We have

$$C_n^2 \int_0^\infty a^3 e^{-\omega a^2} \sum_{p=0}^n \frac{A_p}{(3)_p} \frac{\omega^{\frac{p}{2}} a^p}{p!} \sum_{s=0}^m \frac{A_s}{(3)_s} \frac{\omega^{\frac{s}{2}} a^s}{s!} da = 1, \quad (40)$$

$$C_n^2 \sum_{p=0}^n \sum_{s=0}^m \frac{A_p \omega^{\frac{p}{2}}}{(3)_p p!} \frac{A_s \omega^{\frac{s}{2}}}{(3)_s s!} \int_0^\infty a^{3+p+s} e^{-\omega a^2} da = 1. \quad (41)$$

For $p = s$, $\mu = \nu$, and $m = n$, we get

$$C_n^2 \sum_{p=0}^n \frac{A_p^2 \omega^p}{[(3)_p p!]^2} \int_0^\infty a^{3+2p} e^{-\omega a^2} da = 1. \quad (42)$$

Let us use now the formula

$$\int_0^\infty y^{\lambda-1} e^{-\eta y^u} dy = \frac{1}{u} \eta^{-\lambda/u} \Gamma\left(\frac{\lambda}{u}\right), \quad (43)$$

so we have

$$\int_0^\infty a^{3+2p} e^{-\omega a^2} da = \frac{1}{2} \omega^{-(2+p)} \Gamma(2+p), \quad (44)$$

and inserting equation (44) into equation (42), we have

$$C_n^2 \sum_{p=0}^n \frac{A_p^2 \omega^p}{[(3)_p p!]^2} \frac{1}{2} \omega^{-(2+p)} \Gamma(2+p) = 1, \quad (45)$$

that is,

$$C_n^2 \sum_{p=0}^n \frac{A_p^2}{[(3)_p p!]^2} \frac{1}{2\omega^2} \Gamma(2+p) = 1. \quad (46)$$

Inserting the constant of integration C_n in equation (46) into equation (38), we have

$$\begin{aligned} \psi_n(a) &= \left[\sum_{p=0}^n \frac{A_p^2}{[(3)_p p!]^2} \frac{1}{2\omega^2} \Gamma(2+p) \right]^{-\frac{1}{2}} \\ &\times a^{\frac{3}{2}} e^{-\frac{\omega}{2}a^2} \text{HeunB}\left(2; 0; 2(n+2); \tilde{\delta}_\mu^n; \omega^{\frac{1}{2}}a\right). \end{aligned} \quad (47)$$

The eigenfunctions are biconfluent Heun functions, satisfying the boundary conditions $\psi_n(0) = 0$ and $\psi_n(\infty) = 0$, and consistent on the interval $[0 + \infty[$. The polynomial condition (34) provides Heun polynomials of the biconfluent case ensuring that the eigenfunctions are well-behaved.

We consider now the case of matter dominated by a negative cosmological constant, so $\Lambda = -|\Lambda|$. The wave functions for the Newtonian universe with negative cosmological constant are given by

$$\begin{aligned} \psi_n(a) &= \left[\sum_{p=0}^n \frac{A_p^2}{[(3)_p p!]^2} \frac{1}{2\omega^2} \Gamma(2+p) \right]^{-\frac{1}{2}} \\ &\times a^{\frac{3}{2}} e^{-\frac{\omega}{2}a^2} \text{HeunB}\left(2; 0; 2(n+2); \tilde{\delta}_\mu^n; \omega^{\frac{1}{2}}a\right), \end{aligned} \quad (48)$$

$$E_n = (n+2)\hbar \left(\frac{|\Lambda|}{3}\right)^{\frac{1}{2}}, \quad n = 0, 1, 2, \dots, \quad (49)$$

with the parameters given by

$$\omega^2 = \frac{|\Lambda|}{3} \frac{m^2}{\hbar^2}; \quad \omega = \left(\frac{|\Lambda|}{3}\right)^{\frac{1}{2}} \frac{m}{\hbar}; \quad \omega^{\frac{1}{2}} = \left(\frac{|\Lambda|}{3}\right)^{\frac{1}{4}} \left(\frac{m}{\hbar}\right)^{\frac{1}{2}}, \quad (50)$$

$$\tilde{\delta} = -4GM \left(\frac{|\Lambda|}{3}\right)^{-\frac{1}{4}} \left(\frac{m}{\hbar}\right)^{\frac{3}{2}}. \quad (51)$$

The eigenvalues are nondegenerate since for each value of n corresponds only one eigenfunction.

5. CONCLUDING REMARKS

We consider a small spherical portion of the universe, where Newton's theory applies and the behavior of this portion may reflect the evolution of the universe. Since the variable $a(t)$ that is referred to as the scale factor is positive, we choose to pay attention to the affine quantization procedures. The solution of the time-independent Schrödinger equation is of type biconfluent Heun function where the two first parameters are fixed to 2 and 0, respectively. Our results are quite similar to the ones in [14] with the difference that performing affine quantization, the Hamiltonian operator gains an extra term that is proportional to $1/a^2$ and that extra term fixes the first parameter of the biconfluent Heun function to the value of 2. In a case of matter dominated by a negative cosmological constant, $\Lambda = -|\Lambda|$, the eigenvalues are positive equally spaced and nondegenerate. Regarding cosmic evolution, the presence of a negative cosmological constant implies that our universe that is a small spherical portion of the universe where Newton's theory applies, has maximal size. The Newtonian approach is much simple from the conceptual and mathematical points of view; however, it met also problems and criticisms [29, 30, 31, 32, 33]. It is interesting to work out under which conditions Newtonian cosmology applies. Recent works in Newtonian cosmology include [34, 35].

APPENDIX: BICONFLUENT HEUN EQUATION

A biconfluent Heun equation, denoted by $BHE(\alpha, \beta, \gamma, \delta)$, is the equation of form

$$xu''(x) + \left[1 + \alpha - \beta x - 2x^2\right] u'(x) + \left\{(\gamma - \alpha - 2)x - \frac{1}{2}(\delta + (\alpha + 2)\beta)\right\} u(x) = 0, \quad (52)$$

in which $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$. It has a regular singular point at 0 and an irregular singular point at ∞ . The biconfluent Heun equation is well known and studied from a mathematical point of view (see [28, 36, 37, 38, 39]) and has some applications in different areas of physics [40, 41, 42]. Referring to [28], pages 203–206, if the biconfluent Heun equation admits a polynomial solution, then, it is necessary that $\gamma - \alpha - 2p = 2n$, where n is some nonnegative integer, holds. When α is not a negative integer, one can denote by $N(\alpha, \beta, \delta, x)$ a power series (analytic) solution that can be written as

$$N(\alpha, \beta, \gamma, \delta, x) = \sum_{p=0}^{\infty} \frac{A_p}{(1+\alpha)_p} \frac{x^p}{p!}, \quad (53)$$

with $(\alpha)_p = \frac{\Gamma(\alpha+p)}{\Gamma(\alpha)}$, $p \geq 0$, and satisfies the three-term recursion formula

$$A_{p+2} = \left\{ (p+1)\beta + \frac{1}{2}[\delta + \beta(1+\alpha)] \right\} A_{p+1} - (p+1)(p+1+\alpha)(\gamma - \alpha - 2 - 2p)A_p = 0, \quad (54)$$

where $A_0 = 1$, $A_1 = \frac{1}{2}[\delta + \beta(1+\alpha)]$. The three-recursion term in equation (54) terminates if and only if $\gamma - \alpha - 2 = 2n$, and $A_{n+1} = 0$ simultaneously where n is some non negative integer. Performing induction, $A_{n+1}(\delta)$ is a polynomial of δ of degree $n+1$ hence possessing at most $n+1$ roots δ_{μ}^n , $\mu = 0, 1, 2, \dots$. When the series solution terminates, then we write

$$P_{n,\mu}(\alpha, \beta; x) = N\left(\alpha, \beta, \alpha + 2(n+1), \delta_{\mu}^n; x\right), \quad 0 \leq \mu \leq n, n = 0, 1, 2, \dots \quad (55)$$

When $\alpha + 1 > 0$ and $\beta \in \mathbb{R}$, then the $(n+1)$ roots are real. When all the roots are simple, then the polynomial solutions described are precisely orthogonal polynomials (see [28], Paragraph 3.3).

CONFLICTS OF INTEREST

The author declares that there are no conflicts of interest regarding the publication of this paper.

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