Dynamical Synchronization, the Horizon Problem, and Initial Conditions for Inflation

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Abstract

We consider the evolution of homogeneous cosmologies toward the future in a dynamical systems formulation. Using a variational equation approach, we show that there is a short period in which transient solutions between the end of a Mixmaster era and a subsequent Friedmannian state exist. Implications of the generic inhomogeneous evolution toward the future, the recollapse problem, the horizon problem, and the initial conditions required for inflation are briefly discussed.

Keywords: cosmology, dynamical systems, initial conditions, mixmaster model, inflation
DOI: 10.31526/LHEP.2023.322

1. INTRODUCTION

The observable part of the universe is at any time comprised of a large number of spatial subregions causally connected to the observer, not all of which are so connected to each other. This, as is well known, creates the so-called horizon problem, why any two subregions in the observer's past causal cone, sufficiently separated as to never had been in causal contact throughout their entire history, now appear synchronized in their physically measurable properties (we shall use the word "sync" for synchronization hereafter). If causally disjoint regions were so delicately brought to sameness very early [1], pp. 525-6, [2], p. 815, [3], p. 506., then the universe was in some very specially synchronized state initially, and the question arises as to why this was so. The basic situation associated with the horizon problem is described in Figure 1, where we see a generic snapshot of an "observer" at G overlooking two spatial points B and E in opposite directions, and the two causally disconnected regions $\mathcal{B} = AC$ and $\mathcal{E} = DF$ placed on the spatial hypersurface Σ.

Inflationary expansion, as is well known, either through a phase transition or more generally, allows for initially "unsynchronized" regions to become homogeneous very early, hence, explaining the subtle differences in the measurements of observables of two such regions [4], Section 4.1B, [5], p. 54, [6], Section 5.1. This explanation of the horizon problem is the result of the causal mechanism of pushing the big bang hypersurface sufficiently earlier, so that the past light cones of the disjoint regions manage to form a nonempty intersection, hence, allowing them to "homogenize" (cf., e.g., [7], Section 9.7.2, [8], Section 28.3). However, even if one accepts inflation as a solution to the horizon problem, there is still the issue of initial conditions for inflation, and the universe must be sufficiently uniform over some large scale before inflation, so that it subsequently inflates. In particular, for large field inflation, one does not have to assume homogeneity over a Hubble domain because inhomogeneities redshift [9], but for small field inflation, there is a problem of initial conditions [10]. Why did the field have a large averaged value suitable for inflation over many domains at the end of the preinflationary era?

This question leads to another one: What is the relation between causality and synchronization at the end of the preinflationary period? During inflation, two regions needed *only* to be able to communicate causally with each other at some specific time in their past history to homogenize and be considered synced.

Here, we introduce the *dynamical synchronization* of spatial domains as a transient or temporal analog of the phase transition used in inflationary models. We show that although causality-induced homogenization may be a sufficient condition for sync, it is by no means a necessary one. In [11, 12], a sync mechanism was introduced as a way to drive the universe to simpler states in the past direction on approach to the initial Belinski-Khalatnikov-Lifshitz ("BKL") singularity [13]. In this paper, however, we focus our attention on the *future* evolution of the universe as it exits an early BKL chaotic phase containing different Mixmaster domains in an inhomogeneous background as conjectured in [14]. In that case, due to an earlier formation of horizons, such regions are causally disconnected. However, we show that they are able to gradually "absorb" each other and for some finite time interval of their future evolution resemble a single Friedmannian domain and so proceed in a symphony (for an analogous effect observed in the context of biological oscillators, see [15, 16, 17]).

The plan of this paper is as follows. In the next section, we establish notation, and show that the evolution equations we use admit the dihedral group as their symmetry group. In Section 3, we develop a variational equation approach, and using it, we show that there is a finite time interval in which an anisotropic Mixmaster domain evolves to become near-Friedmannian in the future direction. In the last section, we present arguments that suggest that the same may happen in the generic inhomogeneous case as the universe exits the BKL phase and enters a subsequent Friedmannian stage, and comment on the possible implications of this model for the recollapse problem, the horizon problem, and the issue of initial conditions for inflation.

2. DIHEDRAL SYMMETRY

The main result of this section is that the equations that describe the evolution of separate Mixmaster domains are invariant under the action of the dihedral group D_3, which is the symmetry group of the equilateral triangle, which consists of plane rotations through $2\pi/3$ angles followed by (the so-

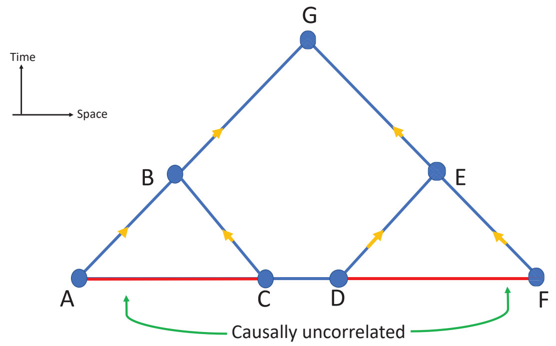

FIGURE 1: According to the horizon problem, rays such as CBG and DEG transport to G the news that the two causally disjoint regions AC and DF are completely synced although never in causal contact.

rotated) x-axis reflections. The main purpose of this section is to describe the path that leads to this result and to establish the notation that will be used subsequently.

Our treatment uses the orthonormal frame formalism for the Einstein equations in which one obtains evolution and constraint equations relative to an orthonormal frame (e_0, e_a), $a = 1, 2, 3$, with e_0 normal to the group orbits, cf. [18], Section 1.4. For the anisotropic Bianchi models of class A, this formalism leads to a dynamical system in further "expansion-normalized" variables. If t denotes proper time and H the Hubble scalar, one introduces the τ-time by $dt/d\tau = 1/H$ and finds equations of the form $\dot{X} = f(X)$, with constraint $g(X) = 0$, where the prime denotes differentiation with respect to the τ-time, f is a polynomial vector field in $\mathbb{R}^n$, and $g(X)$ is a smooth function on $\mathbb{R}^n$ (with n relatively small).

More specifically, the Bianchi cosmologies can be described by a spacetime whose metric admits a 3-dimensional isometry group G_3 acting simply transitively on the spacelike homogeneous hypersurfaces of the spacetime. The Lie algebra of Killing vector fields (with basis ξ_a, and structure constants satisfying $[\xi_a, \xi_b] = C^c_{ab}\xi_c$) is the one associated with the symmetry group G_3. The orthonormal frame spatial vectors commute with the ξ's, $[e_a, \xi_b] = 0$, thus making the orthonormal frame $(\partial/\partial t, e_a)$ "group-invariant". Using the symmetric object η^{mn} and vector a_m, both with constant entries and satisfying $\eta^{mn}a_m = 0$ in our case, the commutation functions $\gamma^c_{ab}(t)$ of the resulting group-invariant, orthonormal frame decompose into the standard form, $\gamma^c_{ab} = \epsilon_{abn}\eta^{mn} + a_a\delta^m_b - a_b\delta^m_a$, with ϵ_{abc} being the alternating symbol with $\epsilon_{123} = +1$. Then, ones arrive at the standard classification of the Bianchi cosmologies into ten different group types using the eigenvalues of the matrix η^{mn} (cf. [18], p. 37).

In the resulting orthonormal frame approach, the metric components $g_{00} = -1$, $g_{0a} = 0$, and the spatial ones are given by the identity matrix, $g_{ab} = \delta_{ab}$, so that the basic gravitational variables are the nonzero commutation functions $\gamma^c_{ab}(t)$, of which now the only remaining ones are as follows: the Hubble scalar H, the diagonal components of the shear tensor σ_{ab} which, being traceless, can be parametrized by only two independent functions σ_+, σ_-, and the three diagonal components of the matrix $\eta_{ab} = \mathrm{diag}(n_1, n_2, n_3)$. Setting $X(\tau) = (N_1, N_2, N_3, \Sigma_+, \Sigma_-)$ where the N's are the normalized spatial scale factors n_a/H, while the Σ's are the normalized σ's, $\sigma_\pm/H$, one obtains the following the evolution equations [19, 18]:

$$N_1' = (q - 4\Sigma_+)\, N_1, \tag{1}$$
$$N_2' = \left(q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-\right) N_2, \tag{2}$$
$$N_3' = \left(q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-\right) N_3, \tag{3}$$
$$\Sigma_+' = -(2-q)\Sigma_+ - 3S_+, \tag{4}$$
$$\Sigma_-' = -(2-q)\Sigma_- - 3S_-, \tag{5}$$

with the following constraint:

$$\Sigma_+^2 + \Sigma_-^2 + \frac{3}{4}\big(N_1^2 + N_2^2 + N_3^2 - 2\left(N_1N_2 + N_2N_3 + N_3N_1\right)\big) = 1, \tag{6}$$

where

$$q = 2\left(\Sigma_+^2 + \Sigma_-^2\right), \tag{7}$$
$$S_+ = \frac{1}{2}\left((N_2 - N_3)^2 - N_1\left(2N_1 - N_2 - N_3\right)\right), \tag{8}$$
$$S_- = \frac{\sqrt{3}}{2}\left(N_3 - N_2\right)\left(N_1 - N_2 - N_3\right). \tag{9}$$

In this section, we prove that the Wainwright-Hsu system (1)–(9) is Γ-equivariant, where Γ is the Σ_+-reflection subgroup of the dihedral group D_3. This then implies that the system is D_3-equivariant.

We begin with some definitions. Let Γ be a compact Lie group, and we consider its action on $\mathbb{R}^n$. We say that the vector field f is Γ-*equivariant* with respect to the action of Γ provided that for each $\gamma \in \Gamma$ and $X \in \mathbb{R}^n$, we have

$$f(\gamma X) = \gamma f(X). \tag{10}$$

In this case, for any solution X of the Γ-equivariant system $\dot{X} = f(X)$, it follows that γX is also a solution.

Now let D_3 be the dihedral group of order 6 acting on the (x, y)-plane $\mathbb{R}^2$. This is the symmetry group of an equilateral triangle with vertices $(1, 2, 3)$ on the plane, where, for concreteness, we assume that the triangle is positioned such that vertex 1 is somewhere on the negative x-axis, and vertices 2 and 3 are on the 4th and 1st quadrants, respectively.

Then, D_3 consists of the three (counterclockwise) rotations r_0, r_1, and r_2 about its center of angles 0, $2\pi/3$, and $4\pi/3$, respectively, and the three reflections s, u, and t, where s is the reflection about the x-axis, that is, $s(123) = 132$, u is the reflection about the axis that forms an angle of $\pi/3$ with the x-axis, that is, $u(123) = 213$, and t is the reflection about the axis that forms an angle of $2\pi/3$ with the x-axis, that is, $t(123) = 321$. Then, $r_0 = I$ is the identity, while the remaining five elements of D_3 can be simply described with the help of the rotation and reflection matrices of the plane.

We now identify the vertices of the equilateral triangle with N_1, N_2, and N_3, and the x and y axes with Σ_+ and Σ_-, respectively. If we set $\Gamma_s = \{I, s\}$, which is the subgroup of D_3 that consists of the identity and the s reflection of D_3, it is not difficult to show that the system (1)–(5) is Γ_s-equivariant. That is, if (N, Σ) is any given solution, then, $s(N, \Sigma)$ is also a solution. This follows by direct calculation (or even by a simple inspection), since when the s element is applied and we get $s(N_1N_2N_3) = N_1N_3N_2$, we also have

$$s\begin{pmatrix}\Sigma_+\\ \Sigma_-\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\Sigma_+\\ \Sigma_-\end{pmatrix}, \tag{11}$$

so that the whole system (1)–(5) is unchanged under s (the constraint (6) is also invariant in that case). (We note that under other subgroups, $\Gamma_t = \{I, t\}$, Γ_u, etc., the system is *not* equivariant.)[1]

The property that the system (1)–(5) and (6) is Γ_s-equivariant is important because of the following reason. From the multiplication table of D_3, it follows that $r_1 \circ u = s = r_2 \circ t$, and therefore, the system is in fact D_3-equivariant, in the sense that it is invariant under $2\pi/3$- and $4\pi/3$-rotations followed by the corresponding x-axis (that is, u or t) reflection (s-reflection corresponds to the element $r_0 \circ s$ of D_3).

D_3-equivariance is in turn important because the D_3-symmetry of rotations and reflections acting on an "initial" (N, Σ) solution as a base is necessary to sustain the BKL sequence of epochs and eras as one progresses to the $t = 0$ singularity in the homogeneous case as developed in [13]. This fact is perhaps not clearly emphasized in the literature, for it is hard to see how the BKL oscillations could be sustained without D_3-equivariance. This symmetry becomes even more important for the BKL conjecture in the inhomogeneous case, where the homogeneous BKL behavior is conjectured to exist all the way to the generic singularity *for each Mixmaster domain*, as discussed in [14, 20]. D_3-symmetry is then valid on any Mixmaster subregion in inhomogeneous spacetime with solutions in distinct subdomains differing only in their spatially dependent phases (while preserving the D_3 symmetry).

3. TRANSIENT BEHAVIOR

We now consider, besides the solution $X = (N, \Sigma)$, another solution $Y = (M, \Pi)$ satisfying the system (1)–(5) and (6), with $p = 2(\Pi_+^2 + \Pi_-^2)$ in the place of q in (7), the spatial curvatures Q's like the S's in (8) and (9) but with the M's in the corresponding places of the N's, and the constraint identical to (6) but with the (M, Π)'s in the places of the (N, Σ)'s.

For τ in some open interval J, and assuming that the solution Y satisfies the initial condition $Y(\tau_0) = Y_0$, we introduce *the synchronization (or variational) function*:

$$\omega = X - Y, \quad \text{on } J, \tag{12}$$

and define *the variational equation along the solution* $Y = (M, \Pi)$:

$$\omega' = D_X f(Y(\tau))\omega = 0, \tag{13}$$

where $D_X f(Y)$ denotes the Jacobian of the vector field $f(X)$ given by the right-hand sides of (1)–(5) evaluated at the solution $Y = (M, \Pi)$. This is a linear equation, with time-dependent coefficients in general.

The basic result associated with the equation (13) is as follows: provided that the initial condition $\omega(\tau_0) = \omega_0$ is small (that is, for X_0 near Y_0), the function $\omega + Y$ is a good approximation to the solution $X = (N, \Sigma)$ on some *compact* interval $J_0 \subset J$ containing τ_0 (for a proof of this result, see [21], pp. 299–302); we note that the *length* of J_0 can be anything provided that J_0 is compact. That is, if $\phi_\tau(X)$ is the flow of the system (1)–(5), and for ξ being small, we consider the spatial derivative of the flow $\partial\phi_\tau(\tau, Y_0)/\partial X = \omega(\tau, \xi)$; then, as ξ tends to zero, the function $Y(\tau) + \omega(\tau, \xi)$ becomes a better and better approximation to $Y(\tau, \xi)$. The former is usually a better choice than the latter, because $\omega(\tau, \xi)$ is linear in ξ. The approximation is uniform in τ on the compact interval J_0 and depends on a smooth variation of the initial condition, $Y_0 \to Y_0 + \xi$ (hence, the name "variational").

Below we shall be interested in the behavior of solutions of the variational equation for two special choices of the particular solution $Y = (M, \Pi)$, both being equilibrium solutions of the system (1)–(5), namely:

(**EQ-1**): *Friedmann-Lemaître point* $\mathcal{F}$:

$$\Sigma_+ = \Sigma_- = 0, \quad N_1 = N_2 = N_3 = 0, \tag{14}$$

(**EQ-2**): *Kasner circle* $\mathcal{K}$,

$$\begin{gathered} \Sigma_+^2 + \Sigma_-^2 = 1, \\ N_1 = N_2 = N_3 = 0, \\ \Sigma_+, \Sigma_-\text{: constants.} \end{gathered} \tag{15}$$

According to the previous theory, for initial conditions X_0 near (**EQ-1**) or (**EQ-2**) (that is, for ω_0 small), the corresponding solutions X of the system (1)–(5) are well approximated by those solutions ω of the variational equation (13) for which the Jacobian is evaluated at the equilibrium solutions (**EQ-1**) and (**EQ-2**), respectively.[2]

Let us consider a homogeneous and anisotropic domain that evolves according to the system (1)–(5).

At the equilibrium (**EQ-1**), the Jacobian $D_X f(Y)$ has eigenvalues $0, 0, 0, -2$, and -2 and corresponding generalized eigenvectors, the standard basis of $\mathbb{R}^5$. This means that, in the (N, Σ) phase space, thought as a plane, every point on the N axis is an equilibrium, and all phase points are stably attracted to the N axis along orbits representing parallel lines to the (vertical) Σ-axis as in Figure 2.

Therefore, on the finite time interval J_0, all solutions of the system (1)–(5) in this case are well approximated by Friedmann universes, the exact types of which depend on the specific point of the N axis on which the orbit lands. All such Friedmann universes have been classified, cf., e.g., [18], p. 129.

To obtain the behavior of the solutions at (**EQ-2**), we introduce the Kasner exponents p_i, $i = 1, 2, 3$, with $\sum p_i = \sum p_i^2 = 1$, where in standard notation in terms of the polar angle ψ, we have: $p_1 = (1 - 2\cos\psi)/3$, $p_{2,3} = (1 + \cos\psi \pm \sqrt{3}\sin\psi)/3$, with $\Sigma_+ = \cos\psi$, $\Sigma_- = \sin\psi$.

[1] The fact that the s reflection is a symmetry of the equations was first noted in [19], cf. their first unnumbered equation in their p. 1416. However, the whole statement in that reference after their equation (2.29) is misleading, because the previously stated symmetry in the last unnumbered equation of p. 1415 as representing a "rotation through $2\pi/3$ rad in the $\Sigma_+ - \Sigma_-$ plane" is erroneous. As presented in that reference, the first transformation that appears in the last line on p. 1415 is the element r_1, while the second one written there is the element r_2 in our notation (in which case one gets a full rotation around the circle, that is, the identity element).

[2] We note here the following important fact. For any equilibrium solution $\bar{Y}$ of the system (1)–(5), the Jacobian $D_X f(\bar{Y})$ is a constant matrix. This matrix, depending on the nature of the equilibrium $\bar{Y}$, may or may not have some eigenvalues on the imaginary axis. Of course, when the equilibrium $\bar{Y}$ is nonhyperbolic, the stability of the solutions of the system (1)–(5) near such an equilibrium cannot be described by studying the linearized equation, and other methods are needed to study the behavior of the solutions of the system (1)–(5). However, the importance of the variational equation in this case is that it does describe the *transient or "observable"* behavior of the solutions of (1)–(5), that is, their behavior on a *compact* time interval (rather than their long-term behavior ($\tau \to \pm\infty$) as implied in stability studies).

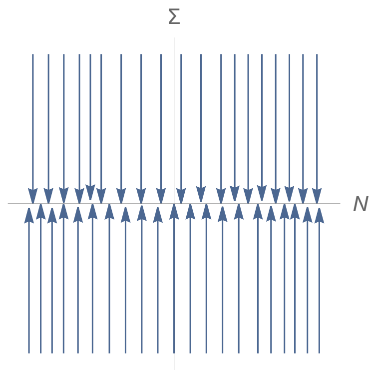

FIGURE 2: Phase portrait giving the behavior of the orbits near the equilibrium (**EQ-1**).

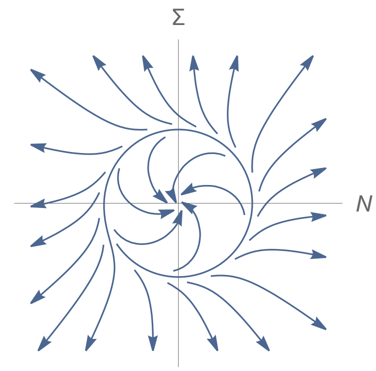

FIGURE 3: Symbolic phase portrait depicting the behavior of the orbits during the transient period from last BKL era (Kasner circle) to the Friedmann state at the origin.

Then, the eigenvalues of the Jacobian $D_X f$(**EQ-2**) are $6p_1, 6p_2, 6p_3, 0, 4$, with corresponding generalized eigenvectors (no relation to the e's of the orthonormal frame previously), and e_1, e_2, e_3, $(0,0,0,-\tan\psi,1)$, $(0,0,0,\cot\psi,1)$, where e_i, $i=1,2,3$, are the first three vectors of the standard basis of $\mathbb{R}^5$. The 0- and 4-eigenspaces are in the (Σ_+, Σ_-)-plane along the directions defined by the two last eigenvectors, while the three eigenspaces corresponding to the remaining three eigenvalues are along the N_1, N_2, and N_3 directions, respectively.

The properties of the solutions near (**EQ-2**) during their transient passage on the time interval J_0 are classified according to various subsets of the Kasner circle. There are two cases of such sets on the Kasner circle:

(1) The special case of the so-called Taub points, where two of the Kasner exponents are zero, namely, the p_i's are $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, respectively. In this case, the eigenvalues become $0,0,0,1,4$, and so due to the unstable directions, the system is repelled from the Kasner circle in the future direction.

(2) In all other cases, all of the p_i's are nonzero, with two of them being positive and one negative. This means that the eigenvalues are $0,-,+,+,4$; that is, there is always a zero eigenvalue corresponding to the eigendirection $(0,0,0,-\tan\psi,1)$, one negative eigenvalue depending on which of the p_i's is negative corresponding to one of the eigendirections N_1, N_2, and N_3, two positive eigenvalues corresponding to the remaining two N_i-directions, and the eigenvalue 4 along the eigendirection $(0,0,0,\cot\psi,1)$.

Because of the presence of zero, negative, and positive eigenvalues, the system is thus generally unstable in the future direction. In particular, we can obtain a four-dimensional unstable set for all solutions of the system (1)–(5) which have the N_{i_0} that corresponds to the zero eigenvalue p_{i_0} equal to zero. The nature of this invariant set depends on the Bianchi type, and, in particular, orbits with two of the N_i's zero exist in that set. Along such orbits (with three zero eigenvalues), the system will move away from the Kasner circle and toward the Friedmann state in the future direction.

Therefore, the behavior of the system resembles the one depicted in the symbolic phase portrait of Figure 3: the system moves away from the Kasner circle (**EQ-2**) in the future because of its instability, and an exit from the phase of the BKL oscillations is realized in the future direction where the system is attracted by the equilibrium (**EQ-1**).

This completes the description of the dynamics of the system during the transient period predicted by the variational equation (13).

4. DISCUSSION

There are various possible implications of these results that we discuss here. One is related to the recollapse problem, another to the horizon problem, and a third one to the impossibility of small field inflation.

Our results show that in the future direction at the end of the BKL period, as a Mixmaster domain evolves off one of the arcs of the Kasner circle of (**EQ-2**) will subsequently be attracted by the Friedmann solution (**EQ-1**) and remain close to it for a finite period of time (given by the "observable" time interval, we denoted earlier by J_0). This describes the mechanism of transient synchronization in a rigorous way (in the future direction and for some finite period of time).

This result brings some new light to the known property of the Bianchi IX universes in the vacuum case that there are no such universes that expand *forever* and must recollapse, cf. [22], Theorem 1. This result also provides an alternative mechanism (other than inflation) for the nonpremature recollapse of anisotropic universes until they become isotropic *by transient synchronization* without the need for an inflationary stage. Since it is known that such universes cannot recollapse until they are isotropized by inflation cf. [23], in a sense, our result increases the probability of inflation to occur because it makes the existence of short-lived anisotropic universes that recollapse before inflation less likely. Our result is also to be compared with the long-term behavior of the Bianchi IX model in the case with a positive cosmological constant, cf. [24], although a nonzero λ is not considered here however.

Once one has a Friedmann domain, what may happen in the next phase of evolution (including the "recollapse" problem discussed above, or studies of small fluctuations) is not completely addressed either by inflation or by the standard approach to cosmological dynamical systems. This is because one has to take into account *dispersive* aspects present in the structure of the Friedmann equations, and the novel properties of the universe that are implied in this sense (for more on this, the reader is invited to study [25]).

One may in fact speculate that in an inhomogeneous setting at the end of the last BKL era, each one of the different

Mixmaster regions involved in the BKL inhomogeneous evolution as they exit the BKL stage in the future direction will, by a similar process, be attracted by and become a "subdomain" in the attracting Friedmann universe. Hence, the Friedmann domain will contain many such regions which will be causally disjoint because of the formation of horizons during the earlier BKL stage (cf., e.g., [26]). The corresponding solutions in other domains will evolve with different *phases* on the Kasner circle compared to the specific Mixmaster domain discussed above; however, one expects that such differences will be erased during the transient period of the Friedmannian evolution.

We end up with a transient period from around the end of the BKL stage to the state described by a Friedmannian domain containing causally disjoined subdomains all synchronized to each other and proceeding in a uniform fashion. As we have already discussed, the length of the transient interval J_0 cannot be taken as a stability interval, and so, we expect the time duration of transient synchronization where the system gets close to Friedmann to be generally short. This provides an alternative approach to the horizon problem.

On a different front, transient synchronization as described here may have an important consequence for early time—in particular—small field inflation. All the various Mixmaster subdomains that eventually connect to a Friedmannian domain at the end of the transient period have arisen at the end of the BKL stage as chaotically oscillating regions in the earlier BKL phase in inhomogeneous spacetime. Each one of these Mixmaster subdomains has therefore well-known stochastic properties as described in [27], and consequently, any function defined on the resulting Friedmann domain at the end of the transient period will have its values *randomly* distributed on the set of these Mixmaster subdomains.

If one imagines a scalar field adjoined to the resulting Friedmannian domain, its *initial*, i.e., pre-inflationary, values would thus be expected to be likewise chaotically distributed, pretty much as dictated by the Mixmaster subdomains at the end of the last BKL era. Therefore, the most natural value of the scalar field potential energy *averaged over all Mixmaster subdomains* in the inhomogeneous setup considered at the end of the BKL stage could naturally be a very large one, of the order of M^4_{Planck}, for classically treated domains of a typical minimum size of the order of M^{-1}_{Planck}. If true, this implies that no small field problem of the sort discussed in [9, 10] could actually be possible at the end of the transient synchronization period. Hence, inhomogeneous initial conditions natural for inflation are probably expected to occur due to transient synchronization.

CONFLICTS OF INTEREST

The author declares that there are no conflicts of interest regarding the publication of this paper.

ACKNOWLEDGMENTS

Thanks are due to two anonymous referees for their important comments and suggestions. I am especially grateful to Robert Brandenberger for his helpful remarks and useful correspondence, and also for providing [9]. I thank Jose Mimoso for his warm hospitality at the Institute of Astrophysics, University of Lisbon, and John Miritzis for many useful discussions. This research was funded by RUDN University, scientific project number FSSF-2023-0003.

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