$(1+1)$-Dimensional Scalar Field Theory on q-Deformed Space

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Abstract

We study scalar field theory in a one-space and one-time dimension on a q-deformed space with a static background. We write the Lagrangian and the equation of motion and solve it to first order in $q-1$, where q is the deformation parameter of the space.

Keywords: noncommutative geometry, noncommutative field theory, q-deformations
DOI: 10.31526/LHEP.2023.392

1. INTRODUCTION

The use of noncommutative geometry in string theory was first introduced in [1], where it was shown that the coordinates of the endpoints of strings on D-branes in the presence of a Neveu-Schwartz field are noncommutative. Noncommutative field theories have also been defined, as they can be derived from string theories and have interesting features, as described in [2, 3].

The introduction of noncommutative spacetime in field theory is motivated by the Heisenberg uncertainty principle in quantum mechanics, which states that, at small distance scales, there is a large uncertainty in momentum measurement. This means that energy can reach very high values in a small spatial distance, approaching the Planck scale. However, according to the general theory of relativity, high energy in a small spatial distance creates a black hole, which prevents the position from being fully certain. One of the approaches to reconcile these two phenomena is to introduce noncommutativity in spacetime, which implies nonlocality in the theory. This is explained in [4, 5].

In this paper, we study $(1+1)$-dimensional classical scalar field theory with static spacetime on a q-deformed space. We present both analytical and numerical analyses of the resulting theory. In Section 2, we review some types of noncommutativity on spacetimes and motivate the choice of q-deformation noncommutativity as the subject of the study. In Section 3, we study the scalar field theory on q-deformed spacetime, write the Lagrangian, and deduce the equation of motion. We also truncate the equation of motion to the linear order in $q-1$ and solved the resulting equation. In Section 4, we study the numerical solutions of the truncated equation of motion showing that the solutions grow exponentially with position and time meaning that the equation is unstable. Thus, there are instabilities in the theory. In Section 5, we conclude the study and suggest topics for further research.

2. TYPES OF NONCOMMUTATIVITY

Here, we briefly review three of the most popular types of noncommutativity relations and justify our motivation to use the q-deformation type.

(1) Canonical noncommutativity: it is the simplest type used in the physics literature. It was introduced in [6]. It is defined by imposing the following commutation relations:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{1}$$

where x^μ are the spacetime coordinates and $\theta^{\mu\nu}$ is a constant, anti-symmetric matrix.
The idea of canonical noncommutativity involves smearing the structure of spacetime in a particular way, regardless of the specific mathematical details of the space. In order to incorporate noncommutative geometry capturing the mathematical structures on the manifold, it is necessary to consider more complex forms of noncommutativity beyond just this basic version.

(2) Lie-type noncommutativity: in this case, the coordinates have a Lie algebra structure; i.e., the commutation relations can capture a Lie algebra structure [7]. The commutation relations are given by

$$[x^\mu, x^\nu] = i f^{\mu\nu}_\rho x^\rho, \tag{2}$$

where $f^{\mu\nu}_\rho$ are the structure constants of the defined Lie algebra. However, this type is not useful because Lie structures are rigid; i.e., any small deformation of a Lie algebra is isomorphic to the original Lie algebra.

(3) q-deformations: a solution to the rigidity problem for the Lie algebras is to replace Lie group with a flexible structure called the quantum group [8, 9, 10]. The term quantum group used in this context refers to the deformations of the universal enveloping algebra of a given group; these objects have Hopf algebra structures which are flexible structures unlike Lie groups and algebras.
The commutation relations are given by

$$x^\mu x^\nu = \frac{1}{q} R^{\mu\nu}_{\sigma\tau} x^\sigma x^\tau, \tag{3}$$

where $q \in (0,1]$ is a parameter and $R^{\mu\nu}_{\sigma\tau}$ is the R-matrix of the quantum group defined on the space.
In this space, a Lie algebra is replaced by a noncommutative Hopf algebra with deformation parameter q. The resulting space is deformed according to the Lie group on the space and on the parameter q; this is the simplest way to deform a spacetime while capturing the full algebraic structure of the space.

3. LAGRANGIAN AND THE EQUATION OF MOTION

We begin with the Lagrangian of the scalar field on the commutative manifold and then introduce noncommutativity by replacing the derivatives with Jackson derivatives [11]. To deform a field theory, we deform the ambient manifold and the fields; the latter manifests in the deformation of its symmetry group. Since the symmetry group is $U(1)$, the deformation of its universal enveloping algebra gives a commutative algebra. Thus, we do not have to worry about defining a product of functions on the new space. The Lagrangian for the commutative theory is

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2. \tag{4}$$

Performing similar calculations as [12, 13] on the q-deformed space, the Lagrangian is

$$\mathcal{L}_q = \frac{1}{2}D_{\mu q}\phi D^\mu_q\phi - \frac{1}{2}m^2\phi^2, \tag{5}$$

where D^μ_q is the Jackson derivative also known as the q-derivative [11] with respect to x_μ, $\mu = 0, 1$ with $x^0 = t$ and $x^1 = x$.

Assuming that the field is defined everywhere and is infinitely differentiable and the deformations are small, i.e., $q \approx 1$, we can relate the theory on the noncommutative topological space to the theory on the commutative manifold by Taylor expanding the q-derivative as a series of ordinary derivatives (i.e., transforming the noncommutative theory back to the commutative manifold). The resulting formulae are

$$D_{xq}(f(x)) = \partial_x f + \sum_{k=1}^{\infty}\frac{(q-1)^k}{(k+1)!}x^k f^{(k+1)}(x), \tag{6}$$

where $f^{(k)}$ is the k-th ordinary derivative of f with respect to x.

$$D_{tq}(f(t)) = \partial_t f + \sum_{k=1}^{\infty}\frac{(q-1)^k}{(k+1)!}t^k f^{[k+1]}(t), \tag{7}$$

where $f^{[k]}$ is the k-th ordinary derivative of f with respect to t.

The resulting Lagrangian on the commutative manifold is

$$\begin{aligned}\mathcal{L}_q = {} & \frac{1}{2}\partial_x\phi\partial_x\phi - \frac{1}{2}m^2\phi^2 + \partial_x\phi\sum_{k=1}^{\infty}\frac{(q-1)^k}{(k+1)!}x^k\phi^{(k+1)} \\ & + \frac{1}{2}\sum_{l,m=1}^{\infty}\frac{(q-1)^{(l+m)}}{(m+1)!(l+1)!}x^{k+l}\phi^{(l+1)}\phi^{(m+1)} + (x\to t),\end{aligned} \tag{8}$$

where $(x \to t)$ means the same terms but with x being replaced by t including in the derivatives.

The Lagrangian is clearly nonlocal as expected from a noncommutative theory. This can be seen from the existence of x^k terms which can relate two points in spacetime arbitrarily far away.

The Lagrangian has an infinite series of derivatives; in this case, the Euler-Lagrange equation will be

$$\frac{\partial\mathcal{L}_q}{\partial\phi} + \sum_{k=1}^{\infty}(-1)^k\frac{d^k}{dx^k}\left(\frac{\partial\mathcal{L}_q}{\partial\phi^{(k)}}\right) + \sum_{k=1}^{\infty}(-1)^k\frac{d^k}{dt^k}\left(\frac{\partial\mathcal{L}_q}{\partial\phi^{[k]}}\right) = 0. \tag{9}$$

The derivatives of the Lagrangian are given by

$$\begin{aligned}\frac{\partial\mathcal{L}_q}{\partial\phi} &= -m\phi, \\ \frac{\partial\mathcal{L}_q}{\partial(\partial\phi)} &= \partial_x\phi + \sum_{n=1}^{\infty}\frac{(q-1)^n}{(n+1)!}x^n\phi^{(n+1)},\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{d}{dx}\left(\frac{\partial\mathcal{L}_q}{\partial(\partial_x\phi)}\right) = {} & \partial_x\partial_x\phi + \sum_{n=1}^{\infty}\left(\frac{n(q-1)^n}{(n+1)!}x^{n-1}\phi^{(n+1)}\right) \\ & + \sum_{n=1}^{\infty}\left(\frac{(q-1)^n}{(n+1)!}x^n\phi^{(n+2)}\right),\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{\partial\mathcal{L}_q}{\partial(\phi^{(k)})} &= \frac{(q-1)^{k-1}x^{k-1}}{k!}\sum_{n=0}^{\infty}\frac{(q-1)^n}{(n+1)!}x^n\phi^{(n+1)}, \\ \frac{d}{dx}\left(\frac{\partial\mathcal{L}_q}{\partial(\phi^{(k)})}\right) & \\ = {} & \frac{(q-1)^{k-1}x^{2k-1}}{k!} \\ & \times\sum_{m,n=0}^{\infty}\binom{k}{m}\frac{(q-1)^n}{(n+1)!}\frac{(n+k+1)!}{(n+2k-m-1)!}x^{n-m}\phi^{(n+k+1)},\end{aligned} \tag{12}$$

with similar formulae for derivatives with respect to t.

Putting all together from (10), (11), and (12) in (9), we get

$$\begin{aligned} & -\partial_x\partial_x\phi - m^2\phi - \sum_{n=1}^{\infty}\frac{n(q-1)^n}{(n+1)!}x^{n-1}\phi^{(n+1)} \\ & - \sum_{n=1}^{\infty}\frac{(q-1)^n}{(n+1)!}x^n\phi^{(n+2)} + \sum_{k=2}^{\infty}(-1)^k\frac{(q-1)^{k-1}x^{2k-1}}{k!} \\ & \times\sum_{n=0}^{\infty}\sum_{m=0}^{k}\binom{k}{m}\frac{(q-1)^n(n+k-1)!}{(n+1)!(n+2k-m-1)!}x^{n-m}\phi^{(n+k+1)} \\ & + (x\to t) = 0.\end{aligned} \tag{13}$$

This is a partial differential equation of infinite order with variable coefficients.

If we consider only small deformations, i.e., $q \approx 1$, then, we can only keep terms up to linear order in $q-1$. The first-order equation will be

$$\begin{aligned} & -\partial_\mu\partial^\mu\phi - m^2\phi \\ & \quad - \frac{(q-1)}{2}\Big[\phi^{(2)} + x\phi^{(3)} - \frac{x^3}{6}\phi^{(3)} \\ & \qquad - x^2\phi^{(3)} - x\phi^{(3)} + (x\to t)\Big] = 0.\end{aligned} \tag{14}$$

This equation is an unstable equation. This may indicate an instability in the theory due to the linear approximation used, but as seen from the full equation of motion, the full theory is stable.

The solution is $\phi = F(t)G(x)$, where

$$\begin{aligned}F(t) &= c_1 e^{iAt/\sqrt{q}} + c_2 e^{-iAt/\sqrt{q}} \\ &+ (q-1)\frac{e^{iAt/\sqrt{q}}}{2iA\sqrt{q}}\left[\frac{iA}{24q}t^4 + \left(\frac{iA^3}{3} - \frac{1}{12\sqrt{q}} - \frac{i}{8A}\right)t^3\right. \\ &\quad + \left(\frac{A+q}{2q} - \frac{A\sqrt{q}}{2} - \frac{i}{8A}\right)t^2 \\ &\quad + \left(\frac{i(A+q)}{2A\sqrt{q}} - \frac{iqA}{4} + \frac{\sqrt{q}}{8A^2}\right)t \\ &\quad \left. + \left(\frac{A+q}{4A^2} + \frac{q\sqrt{A}}{4} + \frac{iq}{16A^3}\right)\right] \\ &+ O\left((q-1)^2\right),\end{aligned} \tag{15}$$

$$\begin{aligned}G(x) &= c_3 e^{ikx/\sqrt{q}} + c_4 e^{-ikx/\sqrt{q}} \\ &+ (q-1)\frac{e^{ikx/\sqrt{q}}}{2ik\sqrt{q}}\left[\frac{ik}{24q}x^4 + \left(\frac{ik^3}{3} - \frac{1}{12\sqrt{q}} - \frac{i}{8A}\right)x^3\right. \\ &\quad + \left(\frac{k+q}{2q} - \frac{k\sqrt{q}}{2} - \frac{i}{8k}\right)x^2 \\ &\quad + \left(\frac{i(k+q)}{2k\sqrt{q}} - \frac{iqk}{4} + \frac{\sqrt{q}}{8k^2}\right)x \\ &\quad \left. + \left(\frac{k+q}{4k^2} + \frac{q\sqrt{k}}{4} + \frac{iq}{16k^3}\right)\right] \\ &+ O\left((q-1)^2\right),\end{aligned} \tag{16}$$

where c_1, c_2, c_3, c_4, and A are normalization constants and $k = \pm\sqrt{A'^2 + m^2}$, where A' is a normalization constant for $G(x)$. When $q = 1$, it reduces to the solution to the Klein-Gordon equation as expected.

4. NUMERICAL RESULTS

Here, we present numerical solutions to the equation of motion to the first order in $q-1$. We focus on $G(x)$ only since the remaining part is similar. The aim is to show that the solutions are exponentially growing and rapidly oscillating establishing that the equation of motion was unstable and the theory has an instability called Ostrogradski instability [14].

We set $c_3 = c_4 = k = 1$, and $A^2 = \frac{1}{2}$ and we plot the solution for different values of the parameter q.

The above results show an instability in the theory leading to divergent solutions to the equations of motion as $x \to \infty$. To remove the instability, we must add infinite terms corresponding to an infinite series of higher derivatives; i.e., we have to consider the full theory or to introduce a degenerate Lagrangian [15], but since the Lagrangian of the full theory is nondegenerate but has infinite series of higher derivatives, it is reasonable to choose the former option. Although the solution for the first-order truncation is unstable, it gives us an intuition on how the q-deformation affects the space. Small q-deformations leading to the fact that nonlocal effects appear to affect the space irregularly with only small effects locally.

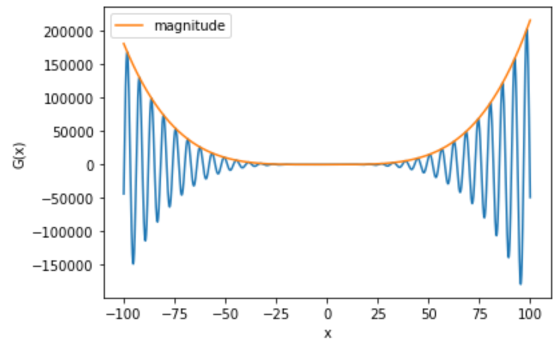

FIGURE 1: At $q-1 = 0.1$, the solution grows exponentially with $|x|$. This is a feature of an unstable equation. In the vicinity of $x = 0$, it is close to the usual Klein-Gordon solution, but as we go further, it becomes more and more distant.

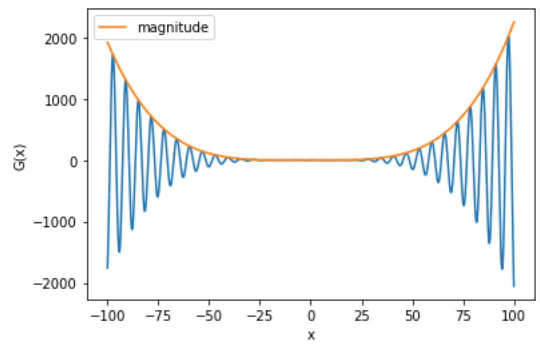

FIGURE 2: At $q-1 = 0.001$, the solution still grows exponentially but slower.

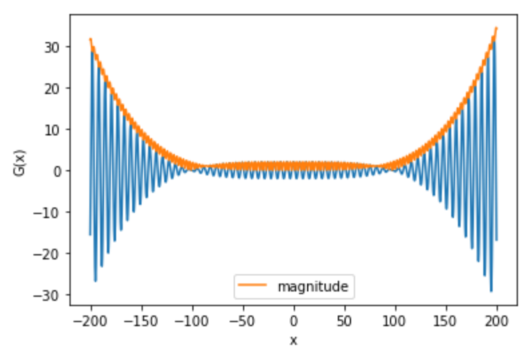

FIGURE 3: At $q-1 = 10^{-6}$, the solution resembles the Klein-Gordon solution up to $|x| = 50$ and then decays for a bit but eventually blows up.

5. CONCLUSION AND OUTLOOK

In conclusion, we showed that defining a field theory on a q-deformed space leads to an infinite series of higher derivatives in the Lagrangian even with a static background. In the case presented, the algebra was commutative, so no new product of functions is needed. We also demonstrated that a truncation to the first order of the theory, and hence any finite order truncation [14, 15], will lead to unstable equations of motion. However, The full theory is stable due to the existence of an infinite series of derivatives.

While we made a progress in the field, much more is to be studied. Future research in this direction should focus on

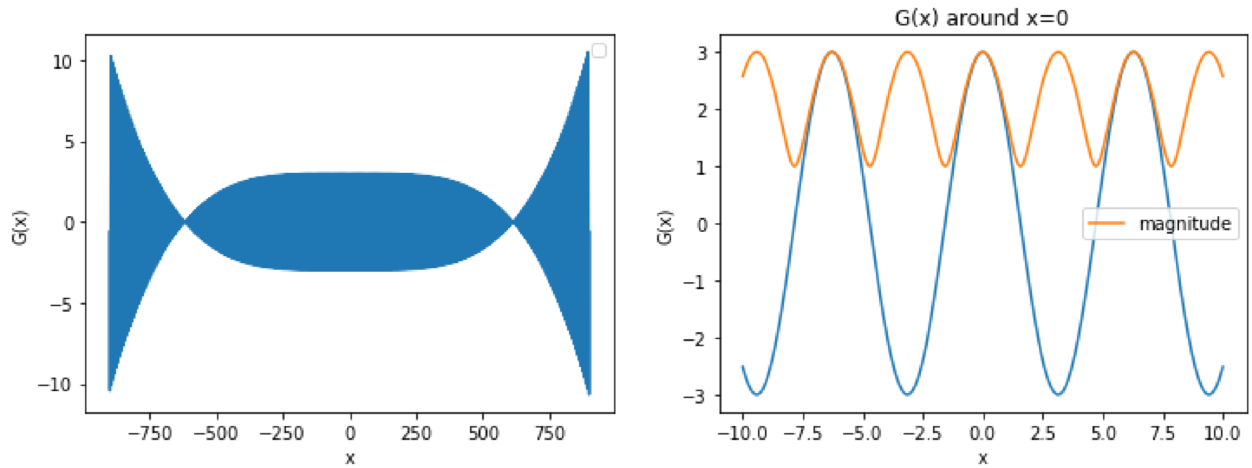

FIGURE 4: The left plot shows that at $q-1=10^{-9}$ the solution has the same behavior as the previous graph but the decay happens at larger $|x|$; all smaller $q-1$ values follow this pattern. The right plot is a plot of $G(x)$ in the vicinity of $x=0$, and the function is a plane wave. This shows that $G(x)$ is approximately a plane wave (a solution for Klein-Gordon equation) at small $|x|$.

defining more complicated theories on q-deformed spaces with noncommutative function algebras and with dynamical spacetimes, also to define higher spin fields on such space and study the new symmetries of the theories as well as the types of instabilities arise if the Lagrangian is truncated.

CONFLICTS OF INTEREST

The author declares that there are no conflicts of interest regarding the publication of this paper.

ACKNOWLEDGMENTS

I would like to thank Dr. Ivan Kolar for the useful discussions on the topic.

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