Macroscopic Noncommutative-Geometry Wormholes as Emergent Phenomena

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Abstract

Noncommutative geometry, an offshoot of string theory, replaces point-like particles with smeared objects. These local effects have led to wormhole solutions in a semiclassical setting, but it has also been claimed that the noncommutative effects can be implemented by modifying only the energy-momentum tensor in the Einstein field equations, while leaving the Einstein tensor unchanged. The implication is that noncommutative-geometry wormholes could be macroscopic. The purpose of this paper is to confirm this conclusion in a simpler and more concrete manner by showing that the throat radius can indeed be macroscopic. This result can be readily explained by considering the noncommutative-geometry background to be a fundamental property and the macroscopic wormhole spacetime to be emergent.

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1. INTRODUCTION

Wormholes are handles or tunnels in spacetime connecting widely separated regions of our Universe or entirely different universes. While wormholes are as good a prediction of Einstein's theory as black holes, they are subject to some severe restrictions. In particular, holding a wormhole open requires a violation of the null energy condition from quantum field theory, calling for the existence of "exotic matter" [1]. This violation is more of a practical than a conceptual problem, as illustrated by the Casimir effect [2]: exotic matter can be made in the laboratory. Being a rather small effect, it may not be sufficient for supporting a macroscopic wormhole. That is the main issue discussed in this paper.

Before continuing, let us recall that the need for exotic matter has always been viewed as a serious barrier for constructing a wormhole. Many attempts have therefore been made to reduce the amount [3] or to eliminate exotic matter entirely by appealing to various modified theories of gravity, such as $f(R)$ or $f(R,T)$ modified gravity, or even by assuming the existence of higher dimensions. For a detailed discussion of this problem, see [4], which deals with wormholes in $4D$ Einstein-Gauss-Bonnet gravity.

Another area dealing with the small effects mentioned above is noncommutative geometry, an offshoot of string theory, where point-like particles are replaced by smeared objects, to be discussed further below. It has been suggested that a noncommutative-geometry background does not prevent a wormhole from being macroscopic. The purpose of this paper is to use the properties of noncommutative geometry to show directly that this is indeed the case. The reason for this outcome is further elaborated on in Section 5: the noncommutative-geometry background is considered to be a fundamental theory, while the resulting macroscopic scale is an emergent phenomenon.

Remark

Noncommutative-geometry wormholes based on the Casimir effect are discussed in [5].

2. BACKGROUND

2.1. Morris-Thorne Wormholes

With the Schwarzschild solution in mind, Morris and Thorne [1] proposed the following static and spherically symmetric line element for a wormhole spacetime:

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1-b(r)/r} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \quad (1)$$

using units in which $c = G = 1$. In the now customary terminology, $\Phi = \Phi(r)$ is called the *redshift function*, which must be finite everywhere to prevent the occurrence of an event horizon. The function $b = b(r)$ is called the *shape function* since it determines the spatial shape of the wormhole when viewed, for example, in an embedding diagram. For our purposes, the most important property is $b(r_0) = r_0$, where $r = r_0$ is the radius of the *throat* of the wormhole. According to [1], the definition of throat requires the *flare-out condition* $b'(r_0) < 1$, while $b(r) < r$ for $r > r_0$ near the throat. The flare-out condition can only be met by violating the null energy condition (NEC)

$$T_{\alpha\beta}k^\alpha k^\beta \geq 0 \quad (2)$$

for all null vectors k^α, where $T_{\alpha\beta}$ is the energy-momentum tensor. Matter that violates the NEC is called "exotic" in [1]. Applied to a wormhole setting, observe that for the radial outgoing null vector $(1,1,0,0)$, the violation reads $T_{\alpha\beta}k^\alpha k^\beta = \rho + p_r < 0$. Here, $T^t{}_t = -\rho(r)$ is the energy density, $T^r{}_r = p_r(r)$ is the radial pressure, and $T^\theta{}_\theta = T^\phi{}_\phi = p_t(r)$ is the lateral (transverse) pressure. Our final requirement is *asymptotic flatness:* $\lim_{r\to\infty}\Phi(r) = 0$ and $\lim_{r\to\infty} b(r)/r = 0$.

For later reference, we now state the Einstein field equations:

$$\rho(r) = \frac{b'}{8\pi r^2}, \quad (3)$$

$$p_r(r) = \frac{1}{8\pi}\left[-\frac{b}{r^3} + 2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}\right], \quad (4)$$

$$p_t(r) = \frac{1}{8\pi}\left(1-\frac{b}{r}\right)\left[\Phi'' - \frac{b'r-b}{2r(r-b)}\Phi' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r-b}{2r^2(r-b)}\right]. \quad (5)$$

2.2. Noncommutative Geometry

The noncommutative-geometry background mentioned in the Introduction is based on the realization that coordinates may become noncommutative operators on a D-brane [6, 7]. A critical feature is that noncommutativity replaces point-like particles with smeared objects [8, 9, 10], thereby eliminating the divergences that are normally unavoidable in general relativity. According to [9], this objective can be realized by showing that spacetime can be encoded in the commutator $[\mathbf{x}^\mu, \mathbf{x}^\nu] = i\theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is an antisymmetric matrix that determines the fundamental cell discretization of spacetime in the same way that Planck's constant $\hbar$ discretizes phase space. According to [11, 12], the smearing can be modeled using a so-called Lorentzian distribution of minimal length $\sqrt{\beta}$ instead of the Dirac delta function: the energy density of a static and spherically symmetric and particle-like gravitational source is given by

$$\rho(r) = \frac{m\sqrt{\beta}}{\pi^2(r^2+\beta)^2}. \tag{6}$$

The usual interpretation is that the gravitational source causes the mass m of a particle to be diffused throughout the region of linear dimension $\sqrt{\beta}$ due to the uncertainty.

Another critical aspect of noncommutative geometry in a wormhole setting is discussed in [9]: it is possible to implement the noncommutative effects in the Einstein field equations $G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$ by modifying only the energy-momentum tensor, while leaving the Einstein tensor $G_{\mu\nu}$ intact. According to [9], the basic reason for this is that a metric field is a geometric structure defined over an underlying manifold whose strength is determined by its curvature, but this is nothing more than the response to the presence of a mass-energy distribution. What is critical here, according to [9], is that noncommutativity is an intrinsic property of spacetime rather than some kind of superimposed structure. So, it has a direct effect on the mass-energy and momentum distributions. The concomitant determination of the spacetime curvature then explains why the Einstein tensor can be left unchanged. As a consequence, the length scales could be macroscopic.

As noted in the Introduction, the purpose of this paper is to confirm this conclusion from the noncommutative-geometry effects in a direct, concrete way. That is the topic of the next section.

3. THE MACROSCOPIC THROAT SIZE

Our first task in this section is to determine the shape function based on equations (3) and (6):

$$\begin{aligned} b(r) &= r_0 + \int_{r_0}^{r} 8\pi(r')^2\rho(r')dr' \\ &= \frac{4m}{\pi}\left[\tan^{-1}\frac{r}{\sqrt{\beta}} - \sqrt{\beta}\frac{r}{r^2+\beta}\right. \\ &\qquad \left. - \tan^{-1}\frac{r_0}{\sqrt{\beta}} + \sqrt{\beta}\frac{r_0}{r_0^2+\beta}\right] + r_0 \\ &= \frac{4m}{\pi}\frac{1}{r}\left[r\tan^{-1}\frac{r}{\sqrt{\beta}} - \sqrt{\beta}\frac{r^2}{r^2+\beta}\right. \\ &\qquad \left. - r\tan^{-1}\frac{r_0}{\sqrt{\beta}} + \sqrt{\beta}\frac{r_0 r}{r_0^2+\beta}\right] + r_0. \end{aligned} \tag{7}$$

Observe that $\lim_{r\to\infty} b(r)/r = 0$; to ensure asymptotic flatness, we retain the assumption $\lim_{r\to\infty}\Phi(r) = 0$. It turns out to be advantageous to let $B = b/\sqrt{\beta}$ be the form of the shape function even though $B(r_0) \neq r_0$. The reason is that B can be expressed as a function of $r/\sqrt{\beta}$ by a simple algebraic rearrangement:

$$\begin{aligned} &\frac{1}{\sqrt{\beta}} b\left(\frac{r}{\sqrt{\beta}}\right) \\ &= B\left(\frac{r}{\sqrt{\beta}}\right) \\ &= \frac{4m}{\pi}\frac{1}{r}\left[\frac{r}{\sqrt{\beta}}\tan^{-1}\frac{r}{\sqrt{\beta}} - \frac{\left(\frac{r}{\sqrt{\beta}}\right)^2}{\left(\frac{r}{\sqrt{\beta}}\right)^2+1}\right. \\ &\qquad \left. - \frac{r}{\sqrt{\beta}}\tan^{-1}\frac{r_0}{\sqrt{\beta}} + \frac{r}{\sqrt{\beta}}\frac{\frac{r_0}{\sqrt{\beta}}}{\left(\frac{r_0}{\sqrt{\beta}}\right)^2+1}\right] + \frac{r_0}{\sqrt{\beta}}; \end{aligned} \tag{8}$$

observe that

$$B\left(\frac{r_0}{\sqrt{\beta}}\right) = \frac{r_0}{\sqrt{\beta}}, \tag{9}$$

the analogue of $b(r_0) = r_0$. Since B is a function of r, we may consider the line element

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{B(r/\sqrt{\beta})}{r/\sqrt{\beta}}} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{10}$$

It now becomes apparent that in view of equation (9), this line element represents a wormhole with throat radius $r_0/\sqrt{\beta}$, while retaining asymptotic flatness.

To check the flare-out condition, let

$$x = \frac{r}{\sqrt{\beta}}, \quad x_0 = \frac{r_0}{\sqrt{\beta}}. \tag{11}$$

Then,

$$\begin{aligned} B(x) &= \frac{1}{\sqrt{\beta}}b(x) \\ &= \frac{4m}{\pi}\frac{1}{r}\left[x\tan^{-1}x - \frac{x^2}{x^2+1} - x\tan^{-1}x_0 + x\frac{x_0}{x_0^2+1}\right] + x_0. \end{aligned} \tag{12}$$

So, $B(x_0) = x_0$, as before, but we also have

$$\begin{aligned} B'(x) &= \frac{1}{\sqrt{\beta}}b'(x) \\ &= \frac{4m}{\pi}\frac{1}{r}\left[\tan^{-1}x + \frac{x}{x^2+1}\right. \\ &\qquad \left. - \frac{2x}{(x^2+1)^2} - \tan^{-1}x_0 + \frac{x_0}{x_0^2+1}\right], \end{aligned} \tag{13}$$

and hence

$$B'(x_0) = \frac{4m}{\pi}\frac{1}{r}\left[\frac{2x_0}{x_0^2+1} - \frac{2x_0}{(x_0^2+1)^2}\right]. \tag{14}$$

Since

$$\frac{x_0}{x_0^2+1} = \frac{\frac{r_0}{\sqrt{\beta}}}{\left(\frac{r_0}{\sqrt{\beta}}\right)^2+1} \to 0 \quad \text{as } \beta \to 0, \tag{15}$$

it follows that

$$B'(x_0) < 1 \quad \text{for } \beta \text{ sufficiently small} \tag{16}$$

independently of m and r. (It is understood that β is always a small nonzero constant.) So, the flare-out condition is met. It also follows that

$$x_0 = \frac{r_0}{\sqrt{\beta}} \text{ is macroscopic.} \tag{17}$$

This is our main conclusion. It confirms the general discussion in Section 2.2 based on the Einstein field equations in conjunction with the noncommutative-geometry background. By taking this to be a fundamental property, the resulting macroscopic scale then becomes an emergent phenomenon. This is discussed further in Section 5.

4. A REMARK ON THE RADIAL TENSION

Given that the radial tension $\tau(r)$ is the negative of the radial pressure $p_r(r)$, it is noted in [1] that equation (4) can be rearranged to yield

$$\tau(r) = \frac{b(r)/r - 2[r-b(r)]\Phi'(r)}{8\pi G c^{-4} r^2}, \tag{18}$$

temporarily reintroducing c and G. From this condition, it follows that the radial tension at the throat is

$$\tau(r_0) = \frac{1}{8\pi G c^{-4} r_0^2} \approx 5 \times 10^{41} \frac{\text{dyn}}{\text{cm}^2}\left(\frac{10\,\text{m}}{r_0}\right)^2. \tag{19}$$

It is also pointed out in [1] that for $r_0 = 3\,\text{km}$, τ has the same magnitude as the pressure at the center of a massive neutron star. Attributing this outcome to exotic matter ignores the fact that exotic matter was introduced to explain the violation of the NEC. It is shown in [13], however, that a noncommutative-geometry background can account for the high radial tension.

5. DISCUSSION AND CONCLUSIONS

The goal of this paper is to obtain a viable model for a macroscopic traversable wormhole by starting with a noncommutative-geometry background involving a particular microscopic effect, the replacement of point-like particles by smeared objects. In the usual terminology, this is an example of a *fundamental property*. Applied to wormholes, one can make use of the following assertions discussed in [9]: noncommutative effects can be implemented in the Einstein field equations by modifying only the energy-momentum tensor, while leaving the Einstein tensor unchanged, thereby implying that the length scales could be macroscopic.

Returning to the smearing effect, this is expressed in equation (6) in mathematical form, now seen as a fundamental property. Equation (6) has led to the shape function $B(x)$, which, in turn, results in a macroscopic throat radius. This outcome is a typical example of an *emergent* property since it does not appear in the fundamental theory. (The concept of emergence dates at least from the time of Aristotle.) By definition, emergent properties or objects are derived from a fundamental theory. Such a process is not reversible, however: in our case, the macroscopic scale does not yield the smearing effect in the original theory. The result is an *effective model* for a macroscopic wormhole precisely because the short-distance effect has been discarded: not only is this information no longer needed, but also it is meaningful only in the fundamental theory. In summary, an effective model for a macroscopic noncommutative-geometry wormhole is necessarily an emergent phenomenon.

CONFLICTS OF INTEREST

The author declares that there are no conflicts of interest regarding the publication of this paper.

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References

[1] M. S. Morris and K. S. and Thorne, *Am. J. Phys.* **56**, 395–412 (1988).
[2] H. G. B. Casimir, *Proc. Kon. Ned. Akad. Wetenschap.* **51**, 793–795 (1948).
[3] P. K. F. Kuhfittig, *Phys. Rev. D* **66**, 024015 (2002).
[4] K. Jusufi, A. Banerjee, and S. G. Ghosh, *Eur. Phys. J. C* **80**, 698 (2020).
[5] P. K. F. Kuhfittig, *J. High Energy Phys. Grav. Cosmol. (JHEPGC)* **9**, 295–300 (2023).
[6] E. Witten, *Nucl. Phys. B* **460**, 335–350 (1996).
[7] N. Seiberg and E. Witten, *J. High Energy Phys.* **9909**, 032 (1999).
[8] A. Smailagic and E. Spallucci, *J. Phys. A* **36**, L-467–L-471 (2003).
[9] P. Nicolini, A. Smailagic, and E. Spallucci, *Phys. Lett. B* **632**, 547–551 (2006).
[10] P. Nicolini and E. Spallucci, *Class. Quant. Grav.* **27**, 015010 (2010).
[11] K. Nozari and S. H. Mehdipour, *Class. Quant. Grav.* **25**, 175015 (2008).
[12] J. Liang and B. Liu, *EPL* **100**, 30001 (2012).
[13] P. K. F. Kuhfittig, *Eur. Phys. J. Plus* **135**, 510 (2020).