# Origin of Internal Symmetries of the Fundamental Interactions, the Family Problem, Fractional Quark Charges, and Unification in the Tangent Bundle Geometry

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## Abstract

In this letter, we follow the hypothesis that the tangent bundle (TB) with the central extended little groups of the $SO(3,1) \rtimes T(1,3)$ group as gauge group is the underlying geometric structure for a unified theory of the fundamental physical interactions. Based on the geometry of the TB, recently, I presented a generalized theory of electroweak interaction in [1]. The vertical (internal) Laplacian of the tangent bundle possesses the same form as the Hamiltonian of a 2D semiconductor quantum Hall system. The three families of leptons and quarks, unlike in the SM, are distinguished by a new quantum number. Here, it will be shown that the $SU(3)$ color symmetry for strong interaction arises as an emergent symmetry similar to Chern-Simon gauge symmetries in multicomponent quantum Hall systems and fractional charge quantization of quarks can be understood by a binding of two vortices to a quark, turning it into a composite quark. The analogy with the anomalous quantum Hall effect could hint at the possible existence of exotic quark states with a hypercharge of $e/5$. Note that based on translational transformations in the TB geometry previously a gauge theoretical understanding of gravity has been achieved. Therefore, the TB can be considered as the underlying geometry that could constitute a possible way for the unification of the known fundamental forces.

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## 1. INTRODUCTION

The standard model (SM) of elementary particles is one of the greatest successes of modern physics. Even though it proved to be extremely successful in explaining a large number of experiments, the SM is incomplete and contains several open problems that cannot be solved on the basis of this model. A fundamental problem is the question of the physical origin of internal symmetry groups. Several other unanswered questions are the mysterious existence of three families of leptons and quarks (also called generations) that differ only by their masses, the lack of understanding of fractional quark charges and of the hierarchy of fermion masses, and the missing explanation of dark matter and dark energy and others.

There have been many attempts to formulate a unified theory beyond the SM that could solve the above puzzles. In Grand Unified Theories, the electroweak and strong interactions are embedded in a larger gauge group, as the $SU(5)$ group first proposed by Georgi and Glashow [2] or in the $SO(10)$ group [3]. There have also been some attempts to understand the origin of families in the SM by using a so-called family symmetry [4, 5]. Despite many subsequent attempts, no unified model exists that is able to solve all these problems on a unified basis and none of them are considered to be universally accepted.

Recently, I proposed following another way to determine the internal symmetries without phenomenological assumptions based on a synthesis of the principle of general relativity and gauge transformations in the frame of a unit underlying geometric structure [1]. The general type of such geometry has been recognized since the 1960s by the discovery of a formal equivalence of gauge theories with the mathematical formalism of fiber bundles [6, 7, 8, 9, 10]. In the fiber bundle interpretation, gauge potentials are understood as a geometric entity (the connection on the principal bundle) and matter fields are described by the associated fiber bundles. In previous studies, the transformation groups of the fibers were taken from the phenomenologically determined gauge groups of the SM. Therefore, the fiber bundle approach mainly delivered a geometrization and reinterpretation of the gauge potentials but could not be used as a bridge to a theory beyond the SM. As a first step in [1], I studied the hypothesis that electroweak interaction is linked with geometric symmetries in the underlying geometric structure of the tangent bundle (TB) with the little groups $SU(2) \otimes E^c(2)$ of the nontransitive group $SO(1,3)$ as structure groups (gauge groups) which includes hypothetical dark matter particles ($E^c(2)$ is the central extended Euklidian group).

In this letter, I study the question of how the color $SU(3)$ symmetry of Quantum Chromodynamics (QCD) can arise in the tangent bundle geometry. Here, the main results of this study are briefly presented, and a full-length paper with additional details and results is published elsewhere [11]. The $SU(3)$ symmetry can not be described as a geometric symmetry of the TB. However, the eigenfunctions of the vertical (internal) Laplacian of the $E^c(2)$ group have the same form as the solution of the 2D Schrödinger equation in real space for electrons in a perpendicular external magnetic field. Combining all tangent fibers at all spacetime points and taking into account the three isospin components of quarks $I_3 = 1/2, 0, -1/2$, the vertical Laplacian of the TB gets the analog form as the multiparticle Hamiltonian of a 2D three-component quantum Hall system [12, 13, 14]. In this approach, emergent effective gauge fields (denoted as Chern-Simon fields) with a local $SU(3)$ symmetry arise in the vertical TB Laplacian in an internal way. An emergent phenomenon is a collective effect of a large number of particles that cannot be deduced from the microscopic theory in a rigorous way. The fractional QHE is a prototype of such phenomenon.

## 2. FUNDAMENTALS OF THE TANGENT BUNDLE GEOMETRY

A tangent bundle associates to every point $x$ of the spacetime manifold $M$ a 4-dimensional tangent space $T_x(M)$ which is the set of all tangent vectors at point $x$. The union of all tangent spaces at all points $x$ of the spacetime manifold $M$ is called the tangent bundle TM. Tangent vectors in $T_x(M)$ can be transformed by the special affine group $G = SO(3,1) \rtimes T(3,1)$. Any point on the TB can be mapped to the base manifold by a projection map $\pi$. In this coordinate description, a point in the tangent bundle is given by the pairs $X = (x, v)$ with $x = (x_0, x_1, x_2, x_3)$ as the coordinate of the spacetime manifold and $v = (v_0, v_1, v_2, v_3)$ as the coordinate of tangent vectors. We introduce tetrad fields $e^\mu_a(x)$ which form an orthonormal basis $g_{\mu\nu}(x)e^\mu_a(x)e^\nu_b(x) = \eta_{ab}$, where $g_{\mu\nu}(x)$ is the metric of the spacetime manifold and $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$ is the metric of the Minkowski space. Each vector $v$ in the coordinate basis can be expressed in the frame basis according to the rule $v^\nu = e^\nu_a(x)v^a$. The scalar product of a vector and a covector is defined as

$$(v, u) = g_{\mu\nu}(x)v^\mu u^\nu = g_{\mu\nu}(x)e^\mu_a(x)e^\nu_b(x)v^a u^b = \eta_{ab}v^a u^b. \quad (1)$$

The scalar product (1) is the governing structure relation of the tangent bundle, and its invariance with respect to certain transformations determines the geometry of the TB. There exist two types of transformations which leave (1) invariant. First, general spacetime coordinate transformations $x^\mu \to y^\mu = y^\mu(x)$ with vectors which are transformed as $v'^\mu(x) = (\partial y^\mu/\partial x^\nu)v^\nu(x)$ do not change the scaler product (1). Besides, at a fixed spacetime point $x$, the tangent vectors can be transformed by a second type of transformation along the tangent vector axis which also preserves the scalar product (1):

$$v'^a = T^a_b(x)v^b + a^a(x), \quad (2)$$

where $T^a_b(x)$ are matrices satisfying the condition $\eta_{ab}T^a_c T^b_d = \eta_{cd}$. Matrix elements $T^a_b(x)$ are elements of the group $SO(3,1)$ describing special linear local transformations with positive determinant depending on the spacetime point $x$ as a parameter. The transformation (2) is determined by the special affine group—the semidirect product group $SO(3,1) \rtimes T(3,1)$ with $T(3,1)$ as the translational group. Poincare transformations and the transformation group (2) for tangent vectors in the TB are described by the same mathematical group. However, both have a principal different geometrical and physical meaning: the first transforms the coordinates of a flat spacetime manifold while the second describes transformations within the tangent fiber $F = T_x(M)$ leaving the spacetime point $x$ unchanged. Therefore, the generators of the translational group $T(3,1)$ are not related to the momentum or the masses of the particles.

The geometry of the TB is closely connected with the conceptual basis of gravity gauge theories. Here, it is assumed that gravity can be described by the teleparallel gravity gauge theory with the gauge group $T(3,1)$ based on the translational connection as a gauge field [16, 17, 18, 19, 20]. In contrast to this theory, we assume that the subgroup $SO(3,1)$ is related to the other fundamental interactions of particle physics and the spin connection does not vanish but is related to the generalized electroweak interaction [1]. The description of matter fields in QFT requires the knowledge of the unitary representations $T_L(g)$ of these groups. Arbitrary transformation by the $SO(3,1)$ group can be expressed in terms of elements of Wigner's little groups and their cosets. Vector representations satisfy the functional equation $T_L(g_1)T_L(g_2) = T_L(g_1g_2)$. These representations contain certain pathologies as, e.g., the Dirac equation for massive particles is not invariant under the Poincare group, but under its universal covering group. Wigner solved this problem by using projective representations of the Poincare group [21]. In quantum theory, the physical symmetry of a group of transformations on a set of vector states has to preserve the transition probability between two states $| \prec \Phi, T_L(g)\Psi \succ |^2 = | \prec \Phi, \Psi \succ |^2$. Therefore generalized representations (denoted as projective representations) are allowed which satisfy the more general composite law: $T_L(g_1)T_L(g_2) = \varepsilon(g_1, g_2)T_L(g_1g_2)$ where $\varepsilon(g_1, g_2)$ is a complex-valued antisymmetric function of the group elements with $|\varepsilon(g_1, g_2)| = 1$ [21, 22]. Any projective representation of a Lie group $G$ is equivalent to the unitary representation of the central extension of the group $G^c$. For the case of simply connected groups like the rotation group $SO(3)$ projective representations are obtained by replacing the group $SO(3)$ by its universal cover $SU(2)$. However, the Euclidean group $E(2)$ is not semi-simple and the covering group $\widetilde{E}(2)$ is not sufficient. One has to use the central extension $E^c(2)$. The group $E^c(2)$ has been studied previously as, e.g., in [22, 23, 24] and consists of elements $(\alpha, \mathbf{a}, \omega)$ with $(\alpha, \mathbf{a}) \in E(2)$, $\omega \in R$. The generators of the central extended $E^c(2)$ group are given by [1]:

$$\begin{aligned} \mathbf{T}^1 &= -i\left(\frac{\partial}{\partial\xi_1} + \frac{1}{2}\xi_2\frac{\partial}{\partial\beta}\right), \\ \mathbf{T}^2 &= -i\left(\frac{\partial}{\partial\xi_2} - \xi_1\frac{1}{2}\frac{\partial}{\partial\beta}\right), \\ \mathbf{T}^3 &= -i\left(\xi_1\frac{\partial}{\partial\xi_2} - \xi_2\frac{\partial}{\partial\xi_1}\right), \\ \mathbf{E} &= -i\frac{\partial}{\partial\beta}, \end{aligned} \quad (3)$$

which satisfy the following commutation rules $[\mathbf{T}^1, \mathbf{T}^2] = i\mathbf{E}$, $[\mathbf{T}^1, \mathbf{T}^3] = -i\mathbf{T}^2$, $[\mathbf{T}^1, \mathbf{T}^3] = -i\mathbf{T}^2$, $[\mathbf{T}^1, \mathbf{T}^3] = -i\mathbf{T}^2$, $[\mathbf{T}^a, \mathbf{E}] = 0$.

The vertical Laplacian of the group $E^c(2)$ is determined by $\mathbf{\Delta}_{E^c} = (\mathbf{T}^1)^2 + (\mathbf{T}^2)^2 + 2\mathbf{T}^3\mathbf{E}$. Using $\xi_1 = \xi\cos\phi$, $\xi_2 = \xi\sin\phi$ and $h_{nm\varkappa}(\xi, \beta, \phi) = \exp(i\varkappa\beta)(\exp(im\phi)g_{nm\varkappa}(\xi)$ for the eigenfunctions of the Laplacian we find with (3):

$$\begin{aligned} &\left[\left(-\frac{1}{\xi}\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi} + \frac{1}{\xi^2}m^2\right) + \varkappa^2\xi^2 - 2\varkappa m\right]g_{nm\varkappa}(\xi) \\ &\quad = \epsilon_{nm\varkappa}g_{nm\varkappa}(\xi). \end{aligned} \quad (4)$$

The solutions of (4) with $h_{nm\varkappa} = N_{nm\varkappa}\exp(i\varkappa\beta)\exp(im\phi)g_{nm\varkappa}$ are given by

$$g_{nm\varkappa} = \left(\exp\left(-\frac{|\varkappa|\xi^2}{2}\right)\left(\sqrt{\varkappa}\xi\right)^{|m|}L^{|m|}_n\left(|\varkappa|\xi^2\right)\right), \quad (5)$$

where $N_{nm\varkappa} = \sqrt{\frac{\varkappa}{\pi}}\left(\frac{n!}{(|m|+n)!}\right)^{\frac{1}{2}}$, $\epsilon_{nm\varkappa} = 4\varkappa(n + \frac{1}{2} + \frac{1}{2}(m + |m|))$, $n = 0, 1, 2, \ldots$, $m = 0, \pm1, \pm2\ldots$, and $m = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$, $\varkappa = \pm1, \pm2, \ldots$, and $\varkappa = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$, $L^{|m|}_n(x)$ are the associated Legendre polynomials and the internal quantum number (IQN) $m$ can be interpreted as a hypercharge known from the SM, but here two additional IQNs arise: the $E^c$ charge $\varkappa$ and the family quantum number $n$ which could elucidate the existence of families in the SM.

The solutions $h_{nm\varkappa}$ in (5) form an orthonormal set and have the analog form like the solutions of the Schrödinger equation in two space dimensions for electrons in a perpendicular external magnetic field $B$ where the entity $\varkappa$ is substituted by $\varkappa \to eB/2$. In the Schrödinger equation, the different levels with quantum number $n = 0, 1, 2, \ldots$ are denoted as Landau levels (LL). Each Landau level is highly degenerate, and the degeneracy is $BA/2\pi$ where $A$ is the area of the system. An important entity in the Quantum Hall effect (QHE) is the filling factor $\nu$ of a Landau level defined as the ratio of the density of electrons $n_{\text{el}}$ to the density of states in a Landau level $n_{\text{deg}} = eB/2\pi$ (in the unit system $c = \hbar = 1$) given by $\nu = 2\pi n_{\text{el}}/eB$. Final dimensional representations of the group $SU(2)$ are here described by complex coordinates $z^1$ and $z^2$ with $|z^1|^2 + |z^2|^2 = 1$. This is equivalent to the usual two-component description by isospinors $(\phi_1(x), \phi_2(x))^T$.

Differential geometry on a fiber bundle can be executed by using the definition of connections and covariant derivatives on the bundle. They can be constructed by the complementary splitting $T_u(P) = H_u(P) \oplus V_u(P)$ of the bundle into a vertical and a horizontal subbundle [25]. In the SM, the vertical subbundle describes the internal degrees of freedom arising by the gauge group. On the other hand, the horizontal subbundle is a concept to formulate the notion of a connection on the fiber bundle. In the geometric structure of fiber bundles, the connection is independent on the metric and can be axiomatically defined as a matrix-valued 1-form [25]. Differentiation along the horizontal lifted curves in the TB is given by the rule $d/d\tau = (dx^\mu/d\tau)D_\mu$, where

$$D_\mu = \frac{\partial}{\partial x^\mu} + ig_1 A^a_\mu(x)\mathbf{J}_a + ig_2 \frac{1}{2} B^a_\mu(x)\mathbf{T}_a + ig_3 C_\mu(x)\mathbf{E} \qquad (6)$$

is the covariant derivative, $\mathbf{J}_a$ (with $a = 1, 2, 3$) are the generators of the $SU(2)$ group, $\mathbf{T}_a$ and $\mathbf{E}$ are the generators of the group $E^c(2)$, and $A^a_\mu(x)$, $B^a_\mu(x)$, and $C_\mu(x)$ are the corresponding connection coefficients (gauge potentials). Here, gravity is neglected. On the TB, one-particle states of the fermion Dirac field labeled by the momentum $\mathbf{p}$ are described by

$$\begin{aligned}\Psi(x, u) = \sum_K \frac{1}{\sqrt{2E_M V}} \Big[ & a_K u_s(p) \chi_M(u) \exp(i\mathbf{px}) \\ & + b^\dagger_K v_s(p) \chi^*_M(u) \exp(-i\mathbf{px}) \Big],\end{aligned} \qquad (7)$$

where the index $s$ characterizes the helicity $s = \{L, R\}$ and $K = \{M, \mathbf{p}, s\}$ with $M = (n, m, \varkappa, j, j_3)$. $a_K(t)$ is the annihilation operator for a particle and $b^\dagger_K(t)$ is the antiparticle creation operator satisfying the anticommutation rules. $E_M$ is the single particle energy and $V$ the volume, and $u_s(p)$ and $v_s(p)$ are the plane wave solutions of the Dirac equation for particles and antiparticles, respectively. $\chi_M(u)$ are the eigenfunctions of the Laplacian of the group $SU(2) \otimes E^c(2)$.

# 3. THE ANALOGY WITH THE FRACTIONAL QUANTUM HALL EFFECT

The Laplacian $\mathbf{\Delta}_{E^c}$ on the group $E^c(2)$ refers to a single tangent fiber at a fixed space point $\mathbf{x}$. Now, we consider the whole space with a finite volume $V = L^3$ and present the fields arranged in a regular cubic lattice of unit cells which are defined by a set of position vectors $\mathbf{R} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3$ in which $n_i = 0, \pm 1, \pm 2$ run over all integers and $\mathbf{a}_i$ are linear independent basis vectors. In the limit $L \to \infty$, we associate a tangent fiber with an elementary cell with volume $\Delta V_i = (\frac{2\pi}{L})^3$. To obtain the vertical Laplacian of the TB, we combine the fibers attached at all space points $x_i$. For the $E^c(2)$ part of the vertical Laplacian of the bundle, we obtain

$$\Lambda_{E^c(2)} = \sum_i \left[ -i\frac{\partial}{\partial \xi_1^{(i)}} - \frac{\varkappa}{2}\xi_2^{(i)} \right]^2 + \left[ -i\frac{\partial}{\partial \xi_2^{(i)}} + \frac{\varkappa}{2}\xi_1^{(i)} \right]^2, \qquad (8)$$

where $\xi_1^{(i)}$ and $\xi_2^{(i)}$ are the corresponding variables of the representation of the $E^c(2)$ group attached to the space cell $i$. The Laplacian (8) has an analog form as the multiparticle Hamiltonian of a 2D quantum mechanical many-electron system in an external uniform magnetic field describing the integer and fractional quantum Hall effect (QHE). In the integer QHE [26], in a 2D layer of a semiconductor at low temperature and strong magnetic fields, the Hall conductance takes quantized values of $\sigma_{xy} = e^2\nu/2\pi\hbar$, where $\nu$ is precisely an integer number, $\nu = 1, 2, \ldots$. In the fractional QHE [27], $\nu$ is not only restricted to take integer values but can take fractional values. The integer QHE can be understood because the 2D electron gas forms an incompressible liquid at the filling factors $\nu = n = 1, 2, 3$ due to the Landau level structure with a finite energy gap for all charged excitations. The fractional QHE was first explained by a theory of Laughlin [28]. He proposed a trial ground-state many-body wavefunction in a partially filled Landau level with filling fraction $\nu = 1/(2p + 1)$, $p = 1, 2, 3, \ldots$, which includes strong Coulomb interaction and correlations among the electrons. Since the vertical Laplacian (8) has the same form as the 2D multiparticle Hamiltonian in a quantum Hall system, we can use the analog Laughlin wavefunction $\Phi^L_{\text{vert}}(\eta)$ for the description of the vertical part of the quark wavefunction in the vacuum $n = 0$:

$$\Phi^L_{\text{vert}}(\eta) = \prod_{i \prec j} \left(\eta_i - \eta_j\right)^k \exp\left( -\frac{1}{4l^2_\varkappa} \sum_i |\eta_i|^2 \right), \qquad (9)$$

where $k = 2p + 1$ must be an odd integer for $\Phi^L_{\text{vert}}(\eta)$ to be totally antisymmetric, $\eta_i = \xi_1^{(i)} + i\xi_2^{(i)} = \xi_i \exp(i\varphi_i)$. This wavefunction describes a uniform “density” state with a partially filled lowest TB-LL with filling factor $\nu_{\text{TB}} = 1/(2p + 1)$. Due to a particle-hole symmetry, there exist also states at the filling factor $\nu_{\text{TB}} = 1 - 1/(2p + 1)$ [28]. In particular, for $p = 1$, we find states with the filling factor $\nu_{\text{TB}} = 1/3$ and $\nu_{\text{TB}} = 2/3$. Laughlin wavefunction exhibits an analog form as a 2D plasma. From this analogy, he derived as a key result that the elementary charged excitations of the ground state would be quasiparticles or quasiholes with fractional electric charges $\pm e/k$ (with $k = 2p + 1$). For spin singlets, a generalized wavefunction has been determined by Halperin [29]. Since the vertical Laplacian (8) has the same form as the 2D multiparticle Hamiltonian in a quantum Hall system, we can use the analog Laughlin or Halperin wavefunctions for the description of the vertical part of the quark wavefunction.

According to the Laughlin wavefunction, the fractional quantum Hall effect (FQHE) is a collective behavior of electrons in a 2D semiconductor system in a magnetic field when all highly degenerate electrons are confined to the lowest Landau level. In semiconductors, the complex variables $\eta_i$ are related to the two space coordinates and the odd number $k$ with the orbital angular momenta responsible for the degeneracy. In the wavefunction (9), the variables $\eta_i$ are corresponding variables of the tangent vectors, $k$ is related to the hypercharge, and the magnetic field is substituted by the new internal quantum number denoted as $E_c$ charge. The collective behavior in the internal (vertical) space of quarks described by the TB bundle Laplacian is a novel phenomenon which has no counterpart in the SM.

# 4. COMPOSITE QUARKS WITH FRACTIONAL HYPERCHARGES

The composite fermion picture of Jain [13, 30] is a convenient way to provide an intuitive idea for the fractionally charged quasiparticles. This concept can be transferred to the understanding of fractionally charged quarks in TB geometry. When all particles are confined in the lowest TB-LL, the wave function (9) is a polynomial of the complex variable $\eta_i = \xi_i \exp(i\phi_i)$ which has a vortex at the origin because a complete loop around the origin changes $\phi_i$ by $2\pi$. A composite quark is simply envisioned as a bound state of a bare quark carrying two quantized vortices of the many-particle wave function.

A composite quark experiences a reduced IQN $m_{\text{eff}}$ and $\varkappa_{\text{eff}}$. This can be calculated in an analog way as in the case of the anomalous QHE taking into account the magnetic field of the $2p$ attached vortices $B_{\text{vort}} = -2p\rho_0\Phi_0$ pointing antiparallel to the external magnetic field $B$ where $\Phi_0 = h/e$ is the elementary quantum of magnetic flux. Thus, with the effective magnetic field $B_{\text{eff}} = B - 2p\rho_0\Phi_0$, the composite fermions experience a reduced effective magnetic field. Using the analogy of $\varkappa$ with the magnetic field $B$ we can substitute $eB_{\text{eff}} \to 2\varkappa_{\text{eff}}$ and obtain $\varkappa_{\text{eff}} = \varkappa/(2p+1)$.

In the SM, the members arranged in different families of leptons and quarks have identical IQNs and properties except for their masses. In contrast, in the TB, different families can be distinguished by the different family quantum numbers $n = 1, 2, 3$ while the vacuum state carries the TB family quantum number $n = 0$. Rules for the determination of the new IQN $\varkappa$ can be found by the Yukawa interaction Lagrangian which includes additional matrix elements $I^Y_{QHU_R} = \int d\mu \chi_H(u)\chi^*_Q(u)\chi_{U_R}(u)$ and $I^Y_{QH^cD_R} = \int d\mu \chi_{H^c}(u)\chi^*_Q(u)\chi_{D_R}(u)$ with the integration measure $d\mu(u) = d\mu_{SU(2)} d\mu_{E^c}$, $\Phi^c_H = i\sigma_2\Phi_H$ and the notation $Q_n = (U_n, D_n)^T_L, (U_n)_R, (D_n)_R$, where $n = 1, 2, 3$ stands for the different families. The matrix elements are nonzero if the following selection rules are fulfilled: $-m_Q + m_H + m_{D_R} = 0$, $-m_Q + m_{H^c} + m_{U_R} = 0$ and $-\varkappa_Q + \varkappa_H + \varkappa_{D_R} = 0$, $-\varkappa_Q + \varkappa_{H^c} + \varkappa_{U_R} = 0$ with $m_H = -m_{H^c} = 1$.

Now, one can interpret the IQNs of different quarks in the composite quark picture according to its composition by bare quarks and attached TB vortices. Left-handed quarks $(U_L, D_L)$ have the IQN $m_{\text{eff}} = 1/3$ which can be interpreted as excitations of composite holes with two attached TB vortices. Right-handed $D_R$ quarks are isospin singles with $I_3 = 0$ and can be interpreted by the assumption that the lowest level is occupied with isospin singlets. Since every isospin component carries the hypercharge $-1/3$, the hypercharge of the $D_R$ quark is $m_{\text{eff}} = -2/3$. The right-handed $U_R$ quarks with $I_3 = 0$ can be identified as excited hole states with two isospin components each of which carry the hypercharge $m_{\text{eff}} = 2/3$. The total hypercharge of the $U_R$ quarks is $m_{\text{eff}} = 4/3$.

The left-handed leptons $(N, E)_L$ are interpreted as fermions without attached TB vortices with $m = -1$ and a completely filled lowest level $\nu = 1$. Right-handed leptons $E_R$ are isospin singlets with $m = -2$ and filling factor $\nu = 2$.

# 5. EMERGENT CHERN-SIMON $SU(3)$ GAUGE FIELDS ON THE TB

A field-theoretical formalism can be derived that allows a simple understanding of the composite fermion concept. In this "Chern-Simon" approach, a singular gauge transformation is used to map the Hamiltonian of interacting electrons to one of electrons coupled to an additional emergent gauge field [31, 32, 33, 34]. Thus, the fractional QHE has a hidden dynamically generated emergent local $U(1)$ gauge symmetry which is responsible for the binding of vortices to the fermions. In such a way, the Laughlin state with partially filled lowest Landau level is mapped into a completely filled fermionic state of composite fermions.

Quarks can occupy three different isospin states: two states are arranged as a left-handed isospin doublet with $I_3 = 1/2$ and $I_3 = -1/2$ and the third is the right-handed isospin singlet $I_3 = 0$. In the Laplacian of the product group $\mathbf{E}^c(2) \otimes SU(2)$, one can assume that in the ground state the three isospin components are degenerate and equally active. In this case, a finite number of states is available for each orbital state within a degenerate Landau level. Therefore, the system possesses an underlying $SU(3)$ symmetry, and the vertical part of the TB Laplacian shows certain analog features of a semiconductor quantum Hall multicomponent system (see, e.g., [15]) which can be realized by a variety of realizations such as multivalley semiconductors, multi-quantum-well systems or in direct analogy to the use of isospin in the present letter by the use of electron spin. Note that here the isospin does not play a role as internal symmetry but only the pure existence of three components and its possible quantum coherence is requested.

To describe the full dynamics in the TB, we have to include the spacetime depending horizontal part $\Phi_{\text{hor}}$ of the wavefunctions; therefore, the multiplets of quark fields in the TB depend both on the spacetime variables $x$ and on the variables $\boldsymbol{\xi} = (\xi_1, \xi_2)$ of the group $\mathbf{E}^c(2)$. This means that number density $\varrho(x, \boldsymbol{\xi})$ and color $SU_c(3)$ spin density $S_A(x, \boldsymbol{\xi})$ in the ground state $n = 0$ take the following form:

$$\begin{aligned} \varrho(x, \boldsymbol{\xi}) &= \Phi^\dagger_{\text{vert}}(x, \boldsymbol{\xi})\Phi_{\text{vert}}(x, \boldsymbol{\xi}), \\ S_A(x, \boldsymbol{\xi}) &= \Phi^\dagger_{\text{vert}}(x, \boldsymbol{\xi})\boldsymbol{\lambda}_A\Phi_{\text{vert}}(x, \boldsymbol{\xi}), \end{aligned} \tag{10}$$

where $\boldsymbol{\lambda}_A$ are the $SU(3)$ Lie algebra matrices. For simplification, we consider the case of a vacuum state with aligned spin and isospin states. The Chern-Simon formalism for the fractional QHE with an emergent $U(1)$ gauge group has been generalized for the inclusion of spin in the QHE [33, 34]. Using an analog approach, the generalized vertical Laplacian of quarks interacting through the emergent $U_{\rm em}(1)\otimes SU_c(3)$ fields is

$$\begin{aligned}\Lambda_{E^c(2)} &= \mathcal{L}_{\rm CS} \\ &+\Phi^\dagger_{\rm vert}\Bigg[\Big(i\partial_0\Phi - G_0^A - A_0\Big) \\ &\quad+\left(-i\frac{\partial}{\partial\xi^1}+\varkappa\xi^2-\lambda_A G_1^A - A_1\right)^2 \\ &\quad+\left(-i\frac{\partial}{\partial\xi^2}-\varkappa\xi^1-\lambda_A G_2^A - A_2\right)^2\Bigg]\Phi_{\rm vert},\end{aligned} \tag{11}$$

where $G_a^A$ are the $SU_c(3)$ Chern-Simon gauge vector potentials, ($\xi_0 = t$, $a = 0,1,2$, $A = 1-8$), $A_a$ is the emergent $U_{\rm em}(1)$ gauge potentials depending on the spacetime coordinates $x$ as a parameter, and $\mathcal{L}_{\rm CS}$ is the non-Abelian Chern-Simon action of the group $U_{\rm em}(1)\otimes SU_c(3)$:

$$\begin{aligned}\mathcal{L}_{\rm CS} &= \frac{1}{4\pi k_2}\varepsilon^{abc}\left[G_a^A\partial_b G_c^A + \frac{1}{3}f_{ABC}G_a^A G_b^B G_c^C\right] \\ &+\frac{1}{4\pi k_1}\varepsilon^{abc}A_a\partial_b A_c\end{aligned} \tag{12}$$

(compare [33, 34, 35]). Here, we introduced $\partial_a = \partial/\partial\xi^a$, and $k_1$ and $k_2$ are integers defining the topological structure of the model. $\varepsilon^{abc}$ is the antisymmetric 2D Levi-Civita symbol and $f_{AB}^C$ are the structure constants of the $SU(3)$ group. The vertical "magnetic" field strength for the $U_{\rm em}(1)$ group in the $E^c$ manifold is defined as $B = \epsilon^{ab}\partial_a A_b$ and for the $SU(3)$ group $G^A = \epsilon^{ab}(\partial_a G_b^A - f_{ABC}G_a^B G_b^C)$. The equation of motion in the vertical (internal) subspace can be obtained by the variation of the vertical Laplacian $\Lambda_{E^c(2)}$ over $A_0$ given by

$$B = \varepsilon^{ab}\partial_a A_b = 2\pi k_1\varrho. \tag{13}$$

The relation (13) is a constraint for the vertical "density" $\varrho$ which is locally proportional to the emergent vertical "magnetic field" $B$. A simple way to analyze the Chern-Simon approach is to make the mean-field approximation in the gauge $A_0 = 0$. Using an average over the variables $\boldsymbol{\xi}$ of the $E^c$ manifold $\prec \rho \succ = \varrho_0$, the TB Chern-Simons field is smeared out to $\prec B \succ = 2\pi k_1\varrho_0$. The effective field $B_{\rm eff}$ is reduced to

$$B_{\rm eff} = B - \prec B \succ = B - 2\pi k_1\varrho_0. \tag{14}$$

Using the analogy of $\varkappa$ with the magnetic field $B$ ($eB \to 2\varkappa$), we can substitute $eB_{\rm eff} \to 2\varkappa_{\rm eff}$ and find a fractional $E^c$ charge $\varkappa_{\rm eff} = \varkappa/(2p+1$. This relation for $\varkappa_{\rm eff}$ agrees with the composite quark interpretation explained above and also with the rules derived from the Yukawa interaction Lagrangian.

Variation over $G_0^A$ yields the constraint

$$G^A = \epsilon^{ab}\left(\partial_a G_b^A - f^{ABC}G_a^B G_b^C\right) = 2\pi k_2 S^A. \tag{15}$$

The invariant internal Chern-Simon $SU(3)$ color-magnetic field $G^A$ is according to (15) directly proportional to the color $SU(3)$ spin density $S^A$. An important property of the $(2+1)D$ Chern-Simon approach is that the large-scale physics of an incompressible $2D$ system (this means that there is an energy gap above the ground state) is determined purely by the Chern-Simon action $\mathcal{L}_{\rm CS}$ in (12) [31]. Other interaction terms in (11) are short range and invisible in the large-distance scale. This leads to an interesting conclusion. The field equation derived from the pure Chern-Simon action $\mathcal{L}_{\rm CS}$ in (12) is given by $G^A = 0$. Since the internal Chern-Simon color-magnetic field $G^A$ in the large-scale limit of the variables of the $E^c$ group vanishes, we find from (15) for the averaged color $SU(3)$ spin density

$$\prec S^A \succ = 0. \tag{16}$$

The vanishing of the average $\prec S^A \succ$ over the variables $\boldsymbol{\xi}$ implies that the ground state is a color singlet. This could be interpreted as a signature of quark confinement in the TB-QFT. This surprising and unexpected result in the large-scale approximation follows from general universal physical principles in the vertical TB Laplacian independent on the microscopic dynamics of quarks in QCD. Note that similar universal properties in a quantum Hall system, such as the quantized Hall conductivity, are known where the theoretical understanding of physical properties is encoded into the large-scale Chern-Simon Lagrangian and does not involve a detailed understanding of the microscopic quantum mechanics of such systems [31].

# 6. THE CONDENSED VACUUM STRUCTURE IN THE TANGENT BUNDLE

The analog form of the vertical Laplacian with the Hamiltonian of a quantum Hall system induces the hypothesis that the vacuum with the IQN $n = 0$ is completely filled with see leptons and composite see quarks and all higher levels $n = 1,2,3$ are empty. In this picture the background charge of vortices cancels the charge with the opposite sign of the composite quarks. When we add one quark state to the system of the completely occupied ground state, it is placed into a higher energy level $n = 1,2,3$ because of the Pauli exclusion principle and leaves an unoccupied state (hole) in the lowest energy state. This resembles the understanding of an electron hole in a semiconductor crystal lattice. In solid-state physics, an electron hole is simply the absence of an electron from a full valence band. In a similar way, the here-introduced quark-hole is a way to conceptualize the interaction of composite quarks (with $n = 1,2,3$) with the full vacuum state $n = 0$ which leads to a redefinition of antiparticles as holes in the completely filled vacuum state. This new understanding of the vacuum as a completely filled band with the family IQN $n = 0$ differs significantly from the interpretation of the vacuum in QFT as a Fock state with zero particle number. In contrast, it associates a finite particle number density or a chemical potential with the vacuum state. In the presence of attractive interaction by gluon exchange, the vacuum is unstable with respect to the formation of a quark condensate due to the pairing of quarks with antiquarks. This phenomenon shows some analogies with exciton condensation in solid states where pairs of electrons form a condensate due to the weak attractive force between electrons and holes. The possible condensation of excitons has been studied theoretically beginning in the 1960th [36, 37, 38]. Studies in semiconductor bilayer systems have provided experimental evidence for the existence of exciton condensation [39].

To explore the condensed vacuum structure in the TB and the dynamics of quark condensation by quark-antiquark pairing in [11], the gap for quark-antiquark pairing has been calculated in the mean-field approximation of a relativistic Hamiltonian formalism describing the two-body interaction of quarks in the TB by gluon exchange. In the gap equation, the main contribution arises at the Fermi surface $|\mathbf{k}| \simeq \mu$. Using this approximation, we find the gap [11]

$$\Delta(p) = \frac{2}{3} b_{qh}^2 \int_{-\delta}^{\delta} \frac{\mu^2 dk}{\sqrt{\Delta_0^2(k) + (k-\mu)^2}} \times \left[ \frac{3}{2} \ln\left(1 + \frac{8\pi^2}{N_f g^2}\right) + \ln\left(1 + \frac{64\pi\mu}{N_f g^2 |p-k|}\right) \right] \Delta(k) \tag{17}$$

with $k = |\mathbf{k}|$, $p = |\mathbf{p}|$, $b_{qh}^2 = \frac{1}{9} g^2 N_f N_C / \pi^2$ and $\delta$ as a cutoff parameter, $N_C = 3$ is the number of colors, and $N_f$ is the number of flavors. An approximate solution of (17) is given by

$$\Delta(k) = \Delta_0 \sin\left[\frac{2}{3} b_{qh} \ln\left(\frac{c\mu}{2k}\right)\right] \tag{18}$$

with $c = 256(\pi^4/g^5)(2/N_f)^{5/2}$ and

$$\Delta_0 = c\mu \exp\left(-\frac{3\pi^2}{\sqrt{2}g}\right), \tag{19}$$

while the quark condensation parameter $C_q$ is approximately estimated by

$$C_q = \prec \Phi \left| \overline{\Psi_f^a(x,u)} \Psi_f^a(x,u) \right| \simeq -N_g \Delta_0 \frac{\mu_{\text{ren}}^2}{8\pi^2}. \tag{20}$$

With $C_q \simeq -(0.240)^3\,\text{GeV}^3$, we find a vacuum chemical potential $\mu \simeq 0.8\,\text{GeV}$ and a gap $\Delta_0 \simeq 0.2\,\text{GeV}$.

## 7. CONCLUSIONS

This paper is a follow-up of [1] with a study of the hypothesis that the tangent bundle (TB) with the structure group $SO(3,1) \rtimes T(3,1)$ is the underlying geometric structure for a unified theory of the fundamental interactions, explaining their common origin and opening a deeper understanding of the relationship between them. Based on this assumption in [1], a generalized theory of electroweak interaction (including hypothetical Dark Matter particles) with the little groups $G = SU(2) \otimes E^c(2)$ of the $SO(3,1)$ group as gauge group has been presented. The tangent bundle is also the geometric fundament for a gauge theory of gravity based on translational transformations $T(3,1)$ of tangent fibers [16, 17, 18, 19, 20]. The present paper describes a possible way that strong interaction can emerge in the tangent bundle geometry. The group $SU(3)$ cannot be described as a geometrical symmetry in the TB, but this symmetry is hidden in the fundamentals of the tangent bundle geometry arising as an emergent symmetry similar to Chern-Simon gauge symmetries in Quantum Hall systems. This assumption is based on the fact that the vertical Laplacian of the TB has the same form as the multicomponent Hamiltonian of a Quantum Hall system. The eigensolutions of the vertical Laplacian exhibit two additional internal quantum numbers (IQN) which explain the existence of lepton and quark families: the $E^c$ charge $\varkappa$ and the family quantum number $n$. The family quantum number $n$ characterizes different states analogous as Landau levels of electrons in an external magnetic field. The lowest quantum number describes a completely occupied vacuum state (filled with see leptons and see quarks). This means the vacuum state (with $n = 0$) differs from excited states (with $n = 1, 2, 3$) describing valence quarks by a different IQN. The analogy with a quantum Hall system allows using the Laughlin wave function for the description of quarks with fractional hypercharges which can be interpreted as composite quarks formed from bare quarks and two attached hypercharge vortices. Taking into account the three isospin components $I_3 = -1/2, 0, 1/2$, the color $SU(3)$ symmetry arises as an emergent gauge symmetry described by $(2+1)D$ Chern-Simon gauge fields. The field equation of the vertical Laplacian including the emergent Chern-Simon fields implies that in the large-scale limit of the variables of the TB Laplacian the ground state is a color singlet demonstrating a signature of quark confinement. This result follows from the general universal principle in the TB vertical Laplacian independent of the microscopic dynamics of quarks in QCD. Besides, in the TB geometry, a new understanding is introduced for the vacuum as the ground state that is occupied with a condensate of quark-antiquark pairs with finite density (or chemical potential). The gap for quark-antiquark pairing is calculated in the mean-field approximation which allows a numerical calculation of the characteristic parameters of the vacuum such as its chemical potential, the quark condensation parameter, and the vacuum energy.

Recently, at CERN, new exotic particles were observed to have formed as tetraquarks containing two quarks and two antiquarks and pentaquarks containing four quarks and one antiquark (for a review, see, e.g., [40]). The analogy with the anomalous QHE could hint at a possible existence of other types of exotic particles formed from exotic quark states with hypercharges of $e/5$ for up and down quarks and the exotic up quark had an electric charge of $(7/10)e$, while the exotic down quark had charge $(-3/10)e$. Pairs of exotic quarks and exotic antiquarks with the same flavor can form neutral flavorless exotic mesons. In solid-state physics, the real existence of $e/5$ charged quasiparticles has been proven by shot-noise measurements in a quantum Hall system [41].

Since the tangent bundle is also the geometric fundament for a gauge theory of gravity based on translational transformations $T(3,1)$ of tangent fibers [16, 17, 18, 19, 20] and for a generalized theory of electroweak interaction [1], one can identify the TB as the underlying geometric structure for a new type of unified geometrized field theory linked with the geometrization program of physics.

## CONFLICTS OF INTEREST

The author declares that he has no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

## DATA AVAILABILITY

No data was used for the research described in the article.

## References

[1] J. Herrmann, Eur. Phys. J. C 79, 779 (2019), ArXiv:1802.03228v3.

[2] H. Georgi and S. Glashow, Phys. Rev. Lett. 32, 438 (1974).
[3] H. Fritsch and P. Minkowski, Ann. Phys. (N.Y.) 93, 193 (1975).
[4] F. Wilszek and A. Zee, Phys. Rev. Lett.42, 421 (1979).
[5] C. D. Froggatt and H. B. Nielsen, Nucl. Phys. B147, 277 (1979).
[6] E. Lubkin, Ann.Phys. (N.Y.) 23, 233 (1963).
[7] A. Trautman, Rep. Math. Phys. 1, 29 (1970).
[8] W. Drechsler and M. E. Meyer, “Fibre Bundle Techniques in Gauge Theories,” Lecture Notes in Physics, Springer, 1977.
[9] M. F. Atiyah, “The geometry of Yang-Mills fields,” Scuola Normale Superiore, Pisa, Italy (1979).
[10] M. Daniel and C. M. Viallet, Rev. Mod. Phys. 52, 175 (1980).
[11] J. Herrmann, Eur. Phys. J. C 82, 947 (2022), arXiv:2207.14557v2 [hep-th].
[12] O. Heinonen (ed.), “Composite Fermions,” World Scienific (1998).
[13] J. K. Jain, “Composite Fermions,” Cambridge University Press (2009).
[14] Z. F. Ezawa, “Quantum Hall Effects,” World Scientific, 2013.
[15] S. M. Girvin and A. H. MacDonald, arXiv: cond-mat/9505087v1.
[16] K. Hayashi and T. Nakno, Prog. Theor. Phys. 38, 491 (1967).
[17] Y. M. Cho, Phys. Rev. D14, 2521 (1976).
[18] K. Hayashi and T. Shirafuji, Phys. Rev. D19, 3524 (1979).
[19] J. W. Maluf, Annalen Phys. 525, 339 (2013).
[20] R. Aldrovandi and J. G. Pereira, “Teleparallel gravity; an introduction” (Springer, Dordrecht, 2012).
[21] E. P. Wigner, Ann. Math. 40, 149 (1939).
[22] V. Bargman, Ann. Math. 59, 1 (1954).
[23] H. Hoogland, J. Phys. A Math. Gen. 11, 1557 (1978).
[24] J. F. Carina, M. A. del Olmo, and M. Santander, J. Phys. A: Math.Gen. 17, 3091 (1984).
[25] S. Kobayashi and K. Nomizu, “Foundation of Differential Geometry,” Interscience Publishers (1963).
[26] K. V. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. 45, 494 (1980).
[27] C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. 48, 1559 (1982).
[28] R. B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).
[29] B. Halperin, Helv. phys. acta 56, 75 (1983).
[30] J. K. Jain, Phys. Rev. Lett. **63**, 199 (1989).
[31] J. Fröhlich and A. Zee, Nucl. Phys. B364, 517 (1991).
[32] A. Lopez and E. Fradkin, Phys. Rev. B51, 4347 (1995).
[33] A. Balatsky and E. Fradkin, Phys. Rev. B 43, 10622 (1991).
[34] J. Fröhlich, T. Kerler, and P. A. Marchetti, Nucl. Phys. 374, 511 (1992).
[35] E. Witten, Commun. Math. Phys. 121, 351 (1989).
[36] I. V. Keldysh and Y. V. Kopaev, Fiz. Twerd. Tela 6, 2791 (1964).
[37] D. Jerome, T. M. Rice, and W. Kohn, Phys. Rev., 158, 452 (1967).
[38] C. Comte and P. Nozieres, Journal de Physique. 432, 691(7), 1069 (1982).
[39] J. P. Eisenstein and A. N. MacDonald, Nature 432, 691 (2004).
[40] A. Ali, J. S. Lange, and S. Stone, arXiv:1706.00610v2.
[41] M. Reznikov et al., Nature 399, 238 (1999).