Representation of Fermions in the Pati-Salam Model

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Abstract

In this paper, a representation of fermions in the Pati-Salam model is suggested. The semileptonic and beyond standard model flavor changing neutral currents of the Lagrangian in this representation of fermions are discussed. A pair of possible Cabibbo-Kobayashi-Maskawa and Pontecorvo-Maki-Nakagawa-Sakata matrices are defined. An effective Lagrangian for this model is given.

Keywords: Pati-Salam model, fermion matrix, flavor mixing lepton collider
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1. INTRODUCTION

The Pati-Salam model [1] is a grand unified theory (GUT) [1, 2, 3, 4, 5, 6] and has the gauge group structure of $SU(4)_L \times SU(4)_R \times SU(4')$, where $SU(4)_L \times SU(4)_R$ is the chiral flavor gauge group, and $SU(4')$ is the color gauge group. The gauge group structure of the Pati-Salam model is beneficial in several aspects

(i) The minimal simple group $SU(5)$ GUT [3] encounters the issue of proton decay, and the modifications used to address the proton decay problem in $SU(5)$ GUT always encounter issues of naturalness.

(ii) If we use a semisimple group as the GUT gauge group instead of a simple group, the standard model (SM) particles phenomena could be unified with the Pati-Salam gauge group $SU(4)_L \times SU(4)_R \times SU(4')$, where $SU(4)_L \times SU(4)_R$ is the chiral flavor gauge group, and $SU(4')$ is the color gauge group. While the gauge group $SU(2)_L \times SU(2)_R \times SU(4')$ model could be used to reproduce the neutral current (NC) and charge current (CC) weak interaction phenomena, the six flavor fermions and flavor mixing phenomena are difficult to reproduce.

(iii) "Lepton number as the fourth color" [1] is a clean and straightforward assumption when visualizing the fermions from a unified viewpoint.

(iv) The fundamental representations of $SU(4)$ are $\mathbf{4}$, $\mathbf{6}$, and $\bar{\mathbf{4}}$. In a GUT, the fermions always fill in the fundamental representation of a gauge group. We know that fermions have six flavors and four colors, and each fermion has a corresponding antifermion. Thus, fermions (antifermions) can be filled in the Pati-Salam gauge group fundamental representation $\mathbf{4} \times \mathbf{6}$ ($\bar{\mathbf{4}} \times \mathbf{6}$).

(v) Dirac matrices are 4×4 matrices. If we do not add (or reduce) the degrees of freedom by hand, the fermions should fill in the 4×4 matrix.

(vi) The flavor mixing matrices, i.e., Cabibbo-Kobayashi-Maskawa (CKM) and Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrices, could arise naturally from $SU(4)_L \times SU(4)_R \times SU(4')$ Pati-Salam model.

(vii) Pati-Salam model [1] as the flat spacetime limits of the Pati-Salam model in curved spacetime, can be derived from the self-parallel transportation principle of the square root Lorentz manifold [7], which is a pure geometry model. An explicit formulation of sheaf quantization [8, 9, 10, 11, 12, 13] on square root Lorentz manifold is given, the relation between sheaf quantization and path integral quantization is shown [14], and the canonical quantization of Yang-Mills theory in curved spacetime which inspired by sheaf quantization can be seen [15] also. The abstract category structure of sheaf quantization of square root Lorentz manifold is almost like Lagrangian submanifold on symplectic geometry [16, 17].

Gauge group structure $SU(4)_L \times SU(4)_R \times SU(4')$ of the Pati-Salam model is the starting point of this paper. In an existing paper [1], the chiral flavor group $SU(4)_L \times SU(4)_R$ degenerates into the chiral group $SU(2)_L \times SU(2)_R$ and reproduces the NC and CC weak interactions transported by Z and $W^{\pm}$ weak gauge bosons, respectively. The left-right symmetry of the Pati-Salam model predicts the existence of right handed neutrinos. The $SU(4')$ color group from the conjecture "lepton number as the fourth color" contains $SU(3')$ quantum chromodynamics (QCDs) and exotic semileptonic processes transported by X bosons. The semileptonic processes preserve *B-L* symmetry and violate baryon lepton number conservation. Topics such as *B-L* symmetry [18, 19], baryogenesis [20, 21, 22, 23, 24, 25, 26, 27, 28, 29], leptogenesis [26, 30, 31, 32, 33, 34, 35], left-right symmetry [36], and right-handed neutrinos [37] have been important topics in theoretical and experimental high energy physics for decades. Recent literature has discussed the flavor violation [38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52], neutral gauge boson [53, 54, 55], lepton quark collider [56], lepton flavor universality [57], gravitational wave imprints [58], muon $g-2$ anomaly [59], and muon collider [60] which relate to the Pati-Salam model and other models.

The original fermion representation in the Pati-Salam model [1] includes only two families of quarks and leptons. In this paper, however, we suggest a representation of fermions in the Pati-Salam model comprising all three families of quark and lepton states as the eigenstates of Lagrangians. We discuss the fermion-antifermion-boson vertex new physics of semileptonic processes transported by X bosons and beyond standard model flavor changing neutral currents (FCNCs) processes transported by neutral bosons Y, based on the novel representation of fermions. We also present a possible construction of the CKM and PMNS matrices based on this representation of fermions. Finally, we illustrate an effective total Lagrangian density for this model.

2. REPRESENTATION OF FERMIONS

The well-established Pati-Salam model [1] has the following gauge group

$$G = SU(4)_L \times SU(4)_R \times SU\left(4'\right), \tag{1}$$

where $SU(4)_L$ and $SU(4)_R$ are the chiral flavor gauge groups, and $SU(4')$ is the color group.

Fermions have six flavors of quarks and leptons. If we gauge the flavor symmetry according to the $SU(6)$ group, the fermions should fill in a 4×6 matrix. The $SU(6)$ flavor symmetry will engage with nine gauge bosons at least that transport flavor gauge interactions. To date, the experimental data only showed us three flavor gauge interaction bosons, which are W^+, W^-, and Z. The problem relates to how to reduce the nine flavor gauge bosons naturally to three, reveal the Standard Model interaction vertices, and reproduce flavor mixing phenomena. Furthermore, it will be hard to reproduce the Gell-Mann-Nishijima formula and flavor mixing phenomena, and the $SU(6) \times SU(4')$ gauge group is not minimal for the GUT.

This $SU(4)_L \times SU(4)_R$ flavor gauge group symmetry restricts the representation matrix of fermions to a 4×4 matrix. For this 4×4 fermion matrix, whether the flavor degrees of freedom will take on the shape of a column or row needs to be established. A minimal coupling Lagrangian is constructed as follows for the color and flavor interaction to answer this question:

$$\mathcal{L} = \mathrm{Tr}\left[i\bar{\Psi}\gamma^\mu\partial_\mu\Psi + f\bar{\Psi}\gamma^\mu V_\mu\Psi - g\bar{\Psi}\gamma^\mu\Psi W_\mu\right], \tag{2}$$

where $f, g \in \mathbb{R}$ are coupling constants. V_μ and W_μ are 4×4 Hermitian matrices and can be decomposed as follows:

$$V_\mu = \sum_{a=1}^{15} V_\mu^a T^a, \quad W_\mu = \sum_{a=1}^{15} W_\mu^a T^a, \tag{3}$$

where T^a ($a = 1, 2, \ldots, 15$) are generators of $SU(4)$ and an example can be found in Appendix A, and V_μ^a and W_μ^a are gauge bosons. The first term in Lagrangian (2) is a kinematic term. The flavor interaction can be chiral decomposed but the color interaction cannot. We observe that the second term in Lagrangian (2) is difficult to decompose due to chiral symmetry, but the third term can be decomposed (the proof is presented in Appendix B) as follows:

$$\mathcal{L} = \mathrm{Tr}\left[i\bar{\Psi}\gamma^\mu\partial_\mu\Psi + \sum_{a=1}^{15}\left(f\bar{\Psi}\gamma^\mu V_\mu^a T^a\Psi - g\bar{\Psi}_L\gamma^\mu\Psi_L W_\mu^a T^a - g\bar{\Psi}_R\gamma^\mu\Psi_R W_\mu^a T^a\right)\right], \tag{4}$$

where the chiral fermions are defined

$$\Psi_L = \frac{1-\gamma^5}{2}\Psi, \quad \Psi_R = \frac{1+\gamma^5}{2}\Psi, \tag{5}$$

$$\bar{\Psi}_L = \Psi^\dagger\frac{1-\gamma^5}{2}\gamma^0, \quad \bar{\Psi}_R = \Psi^\dagger\frac{1+\gamma^5}{2}\gamma^0. \tag{6}$$

Accordingly, the second term in Lagrangian (2) describes the $SU(4')$ color gauge interaction, and the third term in Lagrangian (2) describes the $SU(4)_L \times SU(4)_R$ chiral flavor gauge interaction. We then derive that the column of the 4×4 fermion matrix corresponds to color and the row corresponds to flavor.

Such as "lepton number as the fourth color", it was then easy to fill four colors of fermions, i.e., R, G, B, and L, into the four rows of the fermion matrix. The next approach was to derive how to fill the six flavor fermions into the four columns of the fermion matrix. The six flavor fermions were divided into three families, and each family included two flavor fermions. The action in the path integral formulation of quantum field theory is a phase

$$S = \int d^4x\mathcal{L}, \tag{7}$$

each phase term should with 0-dimension and 0-charge, and the fermion matrix should be result in the model being anomaly free. Consider that the fermions in quantum field theory are the operator-valued field, and the quantum states are the eigenstates of operator-valued field. In quantum mechanics, one operator can correspond to several eigenstates. Then, we suggest a representation of fermions

$$\Psi = \begin{pmatrix} \sqrt{2}u_R & \sqrt{2}c_R & \sqrt{2}t_R & d'_R \\ \sqrt{2}u_G & \sqrt{2}c_G & \sqrt{2}t_G & d'_G \\ \sqrt{2}u_B & \sqrt{2}c_B & \sqrt{2}t_B & d'_B \\ e & \mu & \tau & \nu' \end{pmatrix}, \tag{8}$$

where $C = R, G, B = 1, 2, 3$ are color indices; u, c, and t are the operator-valued fields of three flavor quarks; e, μ, and τ are the operator-valued fields of the electron, mu and tau. Furthermore, ν' and d'_C are the operator-valued fields of neutrinos and d family quarks. Additionally, the $|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle$ neutrino states and $|d_C\rangle, |s_C\rangle, |t_C\rangle$ quark states are eigenstates related to flavor interaction Lagrangian terms containing ν' and d'_C, respectively.

3. GAUGE BOSONS IN THE MINIMAL COUPLING LAGRANGIAN

The possibility of chiral decomposition infers that W_μ^a are gauge bosons transporting flavor gauge interactions, and V_μ^a transporting color gauge interactions. We will discuss the decomposition of the Lagrangian of the flavor and color interactions in detail using the minimal coupling model (2).

3.1. Chiral Flavor $SU(4)_L \times SU(4)_R$ Processes

The gauge boson bears the exchange of quantum numbers charge. For two different fermion-antifermion-boson vertices, when the exchange of the charge is the same, the quantum numbers of two gauge bosons in two fermion-antifermion-boson vertices are the same, except the possibility of mass difference (thanks to the comments from anonymous referees pointing out that even though the quantum numbers of the particles are the same, the masses of the particles might not be the same). The Z boson is a charge-free gauge boson and transports weak NC in the SM, Z boson should be on the diagonal of matrix W_μ, i.e.,

$$Z_\mu = W_\mu^3 = W_\mu^8 = W_\mu^{15}, \tag{9}$$

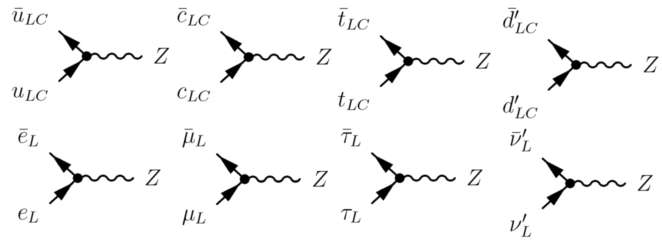

FIGURE 1: In this suggested representation of fermions of the Pati-Salam model, the Lagrangian (11) gives us the fermion-antifermion-boson vertices of weak Z boson interaction with left-handed fermions. For right-handed fermions, the L symbol should be alternated by R.

then the Lagrangian

$$-g\mathrm{Tr}\left[\bar{\Psi}_L\gamma^\mu\Psi_L\sum_{a=3,8,15}W^a_\mu T^a+\{L\to R\}\right] \tag{10}$$

can be decomposed as follows (see Figure 1):

$$\begin{aligned}
&-g\mathrm{Tr}\left[\bar{\Psi}_L\gamma^\mu\Psi_L\sum_{a=3,8,15}W^a_\mu T^a+\{L\to R\}\right]\\
&=-g\mathrm{Tr}\Bigg[\sum_{C=R,G,B}\left(\zeta_1\bar{u}_{LC}\gamma^\mu u_{LC}Z_\mu\right.\\
&\qquad\qquad\left.+\zeta_2\bar{c}_{LC}\gamma^\mu c_{LC}Z_\mu+\zeta_3\bar{t}_{LC}\gamma^\mu t_{LC}Z_\mu\right)\\
&\quad+\frac{1}{2}\left(\zeta_1\bar{e}_L\gamma^\mu e_L Z_\mu+\zeta_2\bar{\mu}_L\gamma^\mu\mu_L Z_\mu\right.\\
&\qquad\left.+\zeta_3\bar{\tau}_L\gamma^\mu\tau_L Z_\mu+\zeta_4\bar{\nu}'_L\gamma^\mu\nu'_L Z_\mu\right)\\
&\quad+\zeta_4\sum_{C=R,G,B}\bar{d}'_{LC}\gamma^\mu d'_{LC}Z_\mu+\{L\to R\}\Bigg],
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
\zeta_1&=1+\frac{\sqrt{3}}{3}+\frac{\sqrt{6}}{6},\quad \zeta_2=-1+\frac{\sqrt{3}}{3}+\frac{\sqrt{6}}{6},\\
\zeta_3&=-\frac{2\sqrt{3}}{3}+\frac{\sqrt{6}}{6},\quad \zeta_4=-\frac{\sqrt{6}}{2}.
\end{aligned} \tag{12}$$

According to the fermion matrix and Lagrangian charge-free assumption, it is easy to find that $W^\pm_\mu$ in this model is

$$W^\pm_\mu=W^9_\mu\pm iW^{10}_\mu=W^{11}_\mu\pm iW^{12}_\mu=W^{13}_\mu\pm iW^{14}_\mu. \tag{13}$$

Furthermore, $W^\pm$ transports the CC in the weak interaction. Then, the Lagrangian, i.e.,

$$-g\mathrm{Tr}\left[\bar{\Psi}_L\gamma^\mu\Psi_L\sum_{a=9}^{14}W^a_\mu T^a+\{L\to R\}\right] \tag{14}$$

can be decomposed as follows (see Figure 2):

$$\begin{aligned}
&-g\mathrm{Tr}\left[\bar{\Psi}_L\gamma^\mu\Psi_L\sum_{a=9}^{14}W^a_\mu T^a+\{L\to R\}\right]\\
&=\frac{-g}{2}\mathrm{Tr}\Bigg[\sqrt{2}\sum_{C=R,G,B}\Big(\bar{u}_{LC}\gamma^\mu d'_{LC}W^+_\mu+\bar{c}_{LC}\gamma^\mu d'_{LC}W^+_\mu\\
&\qquad+\bar{t}_{LC}\gamma^\mu d'_{LC}W^+_\mu+\bar{d}'_{LC}\gamma^\mu u_{LC}W^-\\
&\qquad+\bar{d}'_{LC}\gamma^\mu c_{LC}W^-+\bar{d}'_{LC}\gamma^\mu t_{LC}W^-\Big)\\
&\quad+\bar{e}_L\gamma^\mu\nu'_L W^+_\mu+\bar{\mu}_L\gamma^\mu\nu'_L W^+_\mu+\bar{\tau}_L\gamma^\mu\nu'_L W^+_\mu\\
&\quad+\bar{\nu}'_L\gamma^\mu e_L W^-_\mu+\bar{\nu}'_L\gamma^\mu\mu_L W^-_\mu\\
&\quad+\bar{\nu}'_L\gamma^\mu\tau_L W^-_\mu+\{L\to R\}\Bigg].
\end{aligned} \tag{15}$$

The electric charges of W^+ and W^- are 1 and -1, respectively.

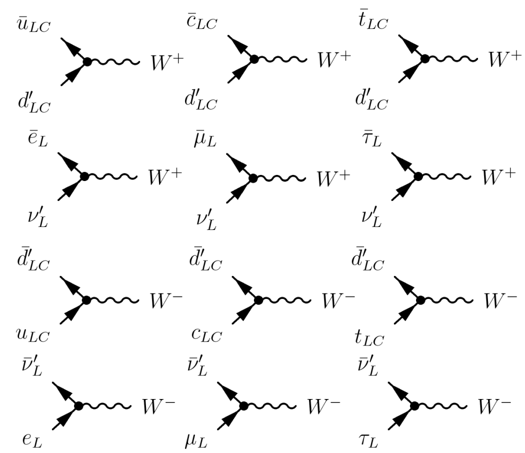

FIGURE 2: The fermion-antifermion-boson vertices of W boson derived by Lagrangian (15), where all three external legs of vertices in this figure are momentum in.

There are new physics chiral flavor processes described by the Lagrangian

$$\begin{aligned}
&-g\mathrm{Tr}\left[\bar{\Psi}_L\gamma^\mu\Psi_L\sum_{a=1,2,4,5,6,7}W^a_\mu T^a+\{L\to R\}\right]\\
&=-g\mathrm{Tr}\Bigg[\sum_{C=R,G,B}\bar{u}_{LC}\gamma^\mu\left(c_{LC}Y^1_{*\mu}+t_{LC}Y^2_{*\mu}\right)\\
&\quad+\frac{1}{2}\bar{e}_L\gamma^\mu\left(\mu_L Y^1_{*\mu}+\tau_L Y^2_{*\mu}\right)\\
&\quad+\frac{1}{2}\bar{\mu}_L\gamma^\mu\left(e_L Y^1_\mu+\tau_L Y^1_{*\mu}\right)\\
&\quad+\sum_{C=R,G,B}\bar{c}_{LC}\gamma^\mu\left(u_{LC}Y^1_\mu+t_{LC}Y^1_{*\mu}\right)\\
&\quad+\sum_{C=R,G,B}\bar{t}_{LC}\gamma^\mu\left(u_{LC}Y^2_\mu+c_{LC}Y^1_\mu\right)\\
&\quad+\frac{1}{2}\bar{\tau}_L\gamma^\mu\left(e_L Y^2_\mu+\mu_L Y^1_\mu\right)+\{L\to R\}\Bigg].
\end{aligned} \tag{16}$$

For example, the predicted beyond SM FCNCs [61, 62, 63]

$$Y_*^1 \to u^C + \bar{c}^C, \tag{17}$$

$$Y_*^1 \to e^+ + \mu^-, \tag{18}$$

have not yet been observed and the mass-generating mechanism of gauge bosons Y^1, Y^2, Y_*^1, and Y_*^2 is interesting. The electric charges of gauge bosons Y^1, Y^2, Y_*^1, and Y_*^2 are 0. The fermion-antifermion-boson vertices about Y are shown in Figure 3. Two examples of beyond SM tree-level FCNCs are in Figure 8.

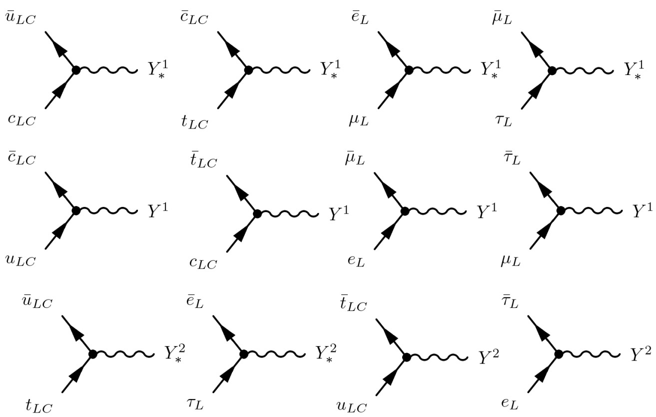

FIGURE 3: The fermion-antifermion-boson vertices of Y are derived by Lagrangian (16), where all three external legs of the vertices are momentum in.

The W_μ matrix is

$$W_\mu = \frac{1}{2}\begin{pmatrix} \zeta_1 Z_\mu & Y_\mu^1 & Y_\mu^2 & W_\mu^- \\ Y_{*\mu}^1 & \zeta_2 Z_\mu & Y_\mu^1 & W_\mu^- \\ Y_{*\mu}^2 & Y_{*\mu}^1 & \zeta_3 Z_\mu & W_\mu^- \\ W_\mu^+ & W_\mu^+ & W_\mu^+ & \zeta_4 Z_\mu \end{pmatrix}. \tag{19}$$

The corresponding electric charge matrix of W_μ is

$$Q_W = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{20}$$

3.2. *Color $SU(4')$ Processes*

We selected V_μ^{15} as the photon. Then, the vertices of the photon from Lagrangian (2) are written as

$$\begin{aligned} & f\mathrm{Tr}\left[\bar{\Psi}\gamma^\mu V_\mu^{15} T^{15}\Psi\right] \\ &= f\frac{\sqrt{6}}{4}\mathrm{Tr}\left[\frac{2}{3}\sum_{C=R,G,B}\left(\bar{u}_C\gamma^\mu V_\mu^{15}u_C + \bar{c}_C\gamma^\mu V_\mu^{15}c_C + \bar{t}_C\gamma^\mu V_\mu^{15}t_C\right)\right. \\ &\qquad - \left(\bar{e}\gamma^\mu V_\mu^{15}e + \bar{\mu}\gamma^\mu V_\mu^{15}\mu + \bar{\tau}\gamma^\mu V_\mu^{15}\tau\right) \\ &\qquad \left. + \frac{1}{3}\sum_{C=R,G,B}\bar{d}'_C\gamma^\mu V_\mu^{15}d'_C - \bar{\nu}'\gamma^\mu V_\mu^{15}\nu'\right]. \end{aligned} \tag{21}$$

Except the neutrinos, the electric charge number preceding each flavor fermion Lagrangian term is correct. As an example, the $\frac{2}{3}$ preceding the Lagrangian term $\bar{u}_C\gamma^\mu V_\mu^{15}u_C$ is the electric charge number of quark u. The experiments show that the neutrino is charge free, such that the neutrino should satisfy the formulas

$$\nu' = e^{i\theta'}, \quad \theta'^\dagger = \theta'. \tag{22}$$

Under the restriction (22), the Lagrangian of neutrinos and photon interaction vertices degenerates into

$$-f\frac{\sqrt{6}}{4}\mathrm{Tr}\left[\bar{\nu}'\gamma^\mu V_\mu^{15}\nu'\right] = -f\frac{\sqrt{6}}{4}\mathrm{Tr}\left[\gamma^0\gamma^\mu V_\mu^{15}\right]. \tag{23}$$

Then, the fermion-antifermion-boson vertices about photon γ on this minimal coupling model are shown in Figure 4.

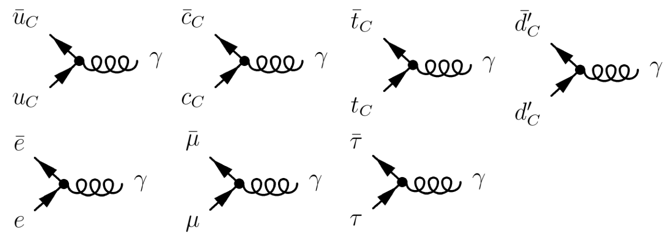

FIGURE 4: The fermion-antifermion-boson vertices of photon derived by Lagrangian (21).

The gauge bosons $V_\mu^1, V_\mu^2, \ldots, V_\mu^8$ are gluons and transport color $SU(3')$ strong interaction and reveal QCD.

There are exotic semileptonic processes [51] transported by $X_\mu^{\pm C}$ particles and the related Lagrangian is

$$\begin{aligned} & f\mathrm{Tr}\left[\bar{\Psi}\gamma^\mu\sum_{a=9}^{14}V_\mu^a T^a\Psi\right] \\ &= f\frac{\sqrt{2}}{2}\sum_{C=R,G,B}\mathrm{Tr}\left[\sqrt{2}\left(\bar{u}_C\gamma^\mu X_\mu^{-C}e + \bar{c}_C\gamma^\mu X_\mu^{-C}\mu + \bar{t}_C\gamma^\mu X_\mu^{-C}\tau\right)\right. \\ &\qquad + \bar{d}'_C\gamma^\mu X_\mu^{-C}\nu' + \bar{\nu}'\gamma^\mu X^{+C}d'_C \\ &\qquad \left. + \sqrt{2}\left(\bar{e}\gamma^\mu X^{+C}u_C + \bar{\mu}\gamma^\mu X^{+C}c_C + \bar{\tau}\gamma^\mu X^{+C}t_C\right)\right], \end{aligned} \tag{24}$$

where

$$X_\mu^{\pm C} = V_\mu^{8+C} \pm iV_\mu^{9+C}. \tag{25}$$

The related fermion-antifermion-boson vertices about X bosons are shown in Figure 5.

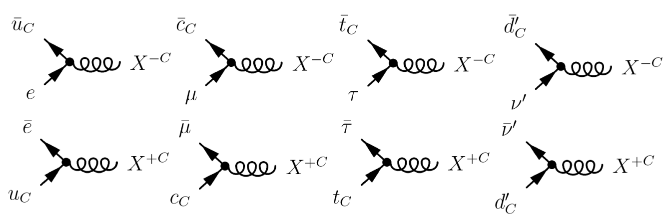

FIGURE 5: The fermion-antifermion-boson vertices derived by Lagrangian (24), where all three external legs of the vertices are momentum in.

The Lagrangian charge-free restriction derives that the charges of X^{-C} and X^{+C} particles are $-\frac{1}{3}$ and $\frac{1}{3}$, respectively. The V_μ matrix is

$$V_\mu = \begin{pmatrix} G_\mu^{RR} + V_\mu^{15} & G_\mu^{RG} & G_\mu^{RB} & X_\mu^{-R} \\ G_\mu^{GR} & G_\mu^{GG} + V_\mu^{15} & G_\mu^{GB} & X_\mu^{-G} \\ G_\mu^{BR} & G_\mu^{BG} & G_\mu^{BB} + V_\mu^{15} & X_\mu^{-B} \\ X_\mu^{+R} & X_\mu^{+G} & X_\mu^{+B} & -3V_\mu^{15} \end{pmatrix}, \tag{26}$$

where $G_\mu^{CC'}$ $(C, C' = R, G, B = 1, 2, 3)$ are gluons and V_μ^{15} is the photon. Then, the electric charge matrix of V_μ is

$$Q_V = \begin{pmatrix} 0 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & -1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{pmatrix}. \tag{27}$$

Three examples of the nonzero semileptonic Feynman diagrams in the tree-level amplitudes are shown in Figure 9, where Figures 9(a) and 9(b) are the t-channel and u-channel of

$$\bar{u}_C + c_C \to e^- + \mu. \tag{28}$$

In addition, Figure 9(c) is the s-channel of the quark lepton interaction

$$c_C + \mu^- \to u_C + e^-. \tag{29}$$

The masses of $X^{\pm C}$ bosons must have been very large because the s, t, and u-channels were still not observed.

The amplitudes in Figure 6 are zero at least on the one-loop level in the model described by Lagrangian (2) because

$$\begin{aligned} M_{\text{total}} \propto & \sum_{a=9}^{14} T_{LC}^a T_{LC}^a - \sum_{a,b=9}^{14} T_{LC}^a T_{CL}^a T_{LC}^b T_{LC}^b \\ & - \sum_{a,b=9}^{14} T_{LC}^a T_{LC}^a T_{CL}^b T_{LC}^b + \cdots \\ = & \, 0 - 0 - 0 + \cdots . \end{aligned} \tag{30}$$

Note that all external fermions in Figure 6 are not antiparticles. The electric charge is not conserved in the process shown in Figure 6 such that the total amplitude $M_{\text{total}} = 0$, which means electric charge conservation avoids quark pair slips to lepton pairs in the minimal coupling model (2).

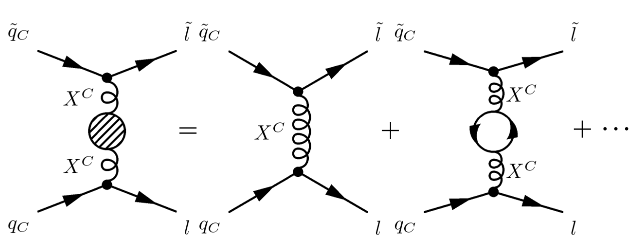

FIGURE 6: The amplitude of the quark pair slips to the lepton pair is zero because of electric charge conservation. The q_C, $\tilde{q}_C$, and $l, \tilde{l}$ are particular quarks and leptons. The vertices in the diagram are described by Lagrangian (2), especially Lagrangian (24).

4. FLAVOR MIXING

The left-handed flavor eigenstates of d, s, and b quark states can be defined as follows:

$$-\frac{\sqrt{2}}{2} g \mathrm{Tr}\left[\bar{u}_{LC}\gamma^\mu d'_{LC} W_\mu^+\right] \left|d'_{LC}\right\rangle = \alpha_1 \left|d'_{LC}\right\rangle, \tag{31}$$

$$-\frac{\sqrt{2}}{2} g \mathrm{Tr}\left[\bar{c}_{LC}\gamma^\mu d'_{LC} W_\mu^+\right] \left|s'_{LC}\right\rangle = \alpha_2 \left|s'_{LC}\right\rangle, \tag{32}$$

$$-\frac{\sqrt{2}}{2} g \mathrm{Tr}\left[\bar{t}_{LC}\gamma^\mu d'_{LC} W_\mu^+\right] \left|b'_{LC}\right\rangle = \alpha_3 \left|b'_{LC}\right\rangle, \tag{33}$$

where $|d'_{LC}\rangle$, $|s'_{LC}\rangle$, and $|b'_{LC}\rangle$ are flavor eigenstates of d, s, and b quarks with left-handed chirality and C color, respectively. The kinematic term of fermions in the Lagrangian (2) is

$$\mathrm{Tr}\left[i\bar{\Psi}\gamma^\mu\partial_\mu\Psi\right] = i\mathrm{Tr}\left[\bar{\Psi}_L\gamma^\mu\partial_\mu\Psi_L + \bar{\Psi}_R\gamma^\mu\partial_\mu\Psi_R\right]. \tag{34}$$

The kinematic term of fermions can be decomposed as follows:

$$\begin{aligned} & i\mathrm{Tr}\left[\bar{\Psi}_L\gamma^\mu\partial_\mu\Psi_L + \bar{\Psi}_R\gamma^\mu\partial_\mu\Psi_R\right] \\ & = i\mathrm{Tr}\Bigg[\sum_{C=R,G,B}\big[2\left(\bar{u}_{LC}\gamma^\mu\partial_\mu u_{LC} + \bar{c}_{LC}\gamma^\mu\partial_\mu c_{LC} + \bar{t}_{LC}\gamma^\mu\partial_\mu t_{LC}\right) \\ & \qquad + \bar{d}'_{LC}\gamma^\mu\partial_\mu d'_{LC}\big] + \bar{e}_L\gamma^\mu\partial_\mu e_L + \bar{\mu}_L\gamma^\mu\partial_\mu\mu_L \\ & \qquad + \bar{\tau}_L\gamma^\mu\partial_\mu\tau_L + \bar{\nu}'_L\gamma^\mu\partial_\mu\nu'_L + \{L \to R\}\Bigg]. \end{aligned} \tag{35}$$

The left-handed mass eigenstates of the d, s, and b quarks are

$$i\mathrm{Tr}\left[\bar{d}'_{LC}\gamma^\mu\partial_\mu d'_{LC}\right]\left|d_{LC}\right\rangle = m_{dL}\left|d_{LC}\right\rangle, \tag{36}$$

$$i\mathrm{Tr}\left[\bar{d}'_{LC}\gamma^\mu\partial_\mu d'_{LC}\right]\left|s_{LC}\right\rangle = m_{sL}\left|s_{LC}\right\rangle, \tag{37}$$

$$i\mathrm{Tr}\left[\bar{d}'_{LC}\gamma^\mu\partial_\mu d'_{LC}\right]\left|b_{LC}\right\rangle = m_{bL}\left|b_{LC}\right\rangle. \tag{38}$$

The CKM matrix is

$$\begin{pmatrix} |d'_{LC}\rangle \\ |s'_{LC}\rangle \\ |b'_{LC}\rangle \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} |d_{LC}\rangle \\ |s_{LC}\rangle \\ |b_{LC}\rangle \end{pmatrix}. \tag{39}$$

The right-handed d, s, and b quark states can be defined after $L \to R$.

Similarly, the left-handed flavor eigenstates of neutrinos are

$$-\frac{1}{2} g \mathrm{Tr}\left[\bar{e}_L\gamma^\mu\nu'_L W_\mu^+\right]\left|\nu_{eL}\right\rangle = \alpha_4\left|\nu_{eL}\right\rangle, \tag{40}$$

$$-\frac{1}{2} g \mathrm{Tr}\left[\bar{\mu}_L\gamma^\mu\nu'_L W_\mu^+\right]\left|\nu_{\mu L}\right\rangle = \alpha_5\left|\nu_{\mu L}\right\rangle, \tag{41}$$

$$-\frac{1}{2} g \mathrm{Tr}\left[\bar{\tau}_L\gamma^\mu\nu'_L W_\mu^+\right]\left|\nu_{\tau L}\right\rangle = \alpha_6\left|\nu_{\tau L}\right\rangle. \tag{42}$$

The left-handed mass eigenstates of neutrinos are

$$i\mathrm{Tr}\left[\bar{\nu}'_L\gamma^\mu\partial_\mu\nu'_L\right]\left|\nu_{1L}\right\rangle = m_{1L}\left|\nu_{1L}\right\rangle, \tag{43}$$

$$i\mathrm{Tr}\left[\bar{\nu}'_L\gamma^\mu\partial_\mu\nu'_L\right]\left|\nu_{2L}\right\rangle = m_{2L}\left|\nu_{2L}\right\rangle, \tag{44}$$

$$i\mathrm{Tr}\left[\bar{\nu}'_L\gamma^\mu\partial_\mu\nu'_L\right]\left|\nu_{3L}\right\rangle = m_{3L}\left|\nu_{3L}\right\rangle. \tag{45}$$

The PMNS matrix is

$$\begin{pmatrix} |\nu_{eL}\rangle \\ |\nu_{\mu L}\rangle \\ |\nu_{\tau L}\rangle \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} |\nu_{1L}\rangle \\ |\nu_{2L}\rangle \\ |\nu_{3L}\rangle \end{pmatrix}. \tag{46}$$

The right-handed eigenstates of neutrinos can be defined similarly after $L \to R$.

5. EFFECTIVE TOTAL LAGRANGIAN AND GAUGE INVARIANCE

An effective total Lagrangian for color, flavor, and Higgs interactions is

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\Big[& i\bar{\Psi}\gamma^\mu\partial_\mu\Psi + f\bar{\Psi}\gamma^\mu V_\mu\Psi - g\bar{\Psi}\gamma^\mu\Psi W_\mu \\ & + \bar{\Psi}\phi\Psi + V(\phi) - \frac{f^2}{2}H^{\mu\nu}H_{\mu\nu} - \frac{g^2\xi}{2}F^{\mu\nu}F_{\mu\nu} \\ & - igF_{\mu\nu}\Psi^\dagger\left(\gamma^\mu\gamma^\nu - \gamma^{\nu\dagger}\gamma^{\mu\dagger}\right)\Psi \\ & + if\Psi^\dagger H_{\mu\nu}\left(\gamma^\mu\gamma^\nu - \gamma^{\nu\dagger}\gamma^{\mu\dagger}\right)\Psi\Big],\end{aligned} \tag{47}$$

where ϕ is the Higgs field; $V(\phi)$ is Higgs potential; $f, g, \xi \in \mathbb{R}$ are coupling constants, and the gauge field strength tensors are

$$H_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - ifV_\mu V_\nu + ifV_\nu V_\mu, \tag{48}$$
$$F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - igW_\mu W_\nu + igW_\nu W_\mu. \tag{49}$$

The second line of Lagrangian (47) represents Yang-Mills theory terms, and the third line is magnetic moment terms. Lagrangian (47) is invariant under local gauge transformations of color space and flavor space rotation $\tilde{U}$ and U, respectively,

$$\Psi' = \tilde{U}\Psi U, \tag{50}$$

where

$$\tilde{U} \in SU(4'), \quad U \in SU(4), \tag{51}$$

such that

$$\gamma^{\mu\prime} = \tilde{U}\gamma^\mu\tilde{U}^\dagger \Rightarrow \gamma^{0\prime}\gamma^{\mu\prime} = \tilde{U}\gamma^0\gamma^\mu\tilde{U}^\dagger, \tag{52}$$
$$V'_\mu = \tilde{U}V_\mu\tilde{U}^\dagger - \left(\partial_\mu\tilde{U}\right)\tilde{U}^\dagger, \tag{53}$$
$$W'_\mu = U^\dagger\left(\partial_\mu U\right) - U^\dagger W_\mu U. \tag{54}$$

6. GAUGE ANOMALY

The Lagrangian (47) is a flat spacetime version of Yang-Mills theory (Pati-Salam type) in curved spacetime and Einstein-Cartan gravity[14]. The curved version theory has deep motivation from the point of view of logic and geometry, which derived from square root metric and self-parallel transportation principle and quantized by sheaf quantization and path integral quantization. The anomaly in quantum field theory always means a symmetry is preserved in classical theory but violated in the quantum version. The global symmetry anomaly might be accessed by quantum field theory, but the local gauge symmetry anomaly (gauge anomaly) is believed to be a consistency condition for a gauge theory. We have to check the anomaly-free condition for the Pati-Salam model with this representation of fermions.

In 4-dimensional spacetime, the quantum gauge anomaly-free condition can be checked by the triangle Feymann diagram in Figure 7.

The amplitude of Figure 7 is proportional to

$$\begin{aligned} iM^{abc\mu\nu\rho} &\propto \mathrm{Tr}\left(T^aT^bT^c\right) + \mathrm{Tr}\left(T^aT^cT^b\right) \\ &= 2\mathrm{Tr}\left(T^aT^{(b}T^{c)}\right) = \frac{1}{2}d^{(abc)}.\end{aligned} \tag{55}$$

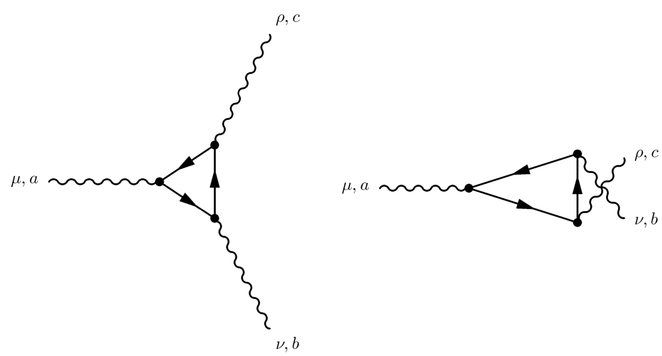

FIGURE 7: The Feynman diagrams about triangle anomaly.

Then, for $SU(4)_L \times SU(4)_R$ chiral Yang-Mills theory, the current conservative equation has the formulation

$$\partial_\mu J^{\mu,a}(x) \propto d^{(a[bc])}\epsilon^{[\mu\nu\rho\sigma]}F^b_{\mu\nu}F^c_{\rho\sigma}, \tag{56}$$

such that the indices bc satisfy commutation and anticommutation relations

$$d^{(a[bc])} = \frac{1}{4}\left(d^{abc} - d^{acb} + d^{acb} - d^{abc}\right) = 0. \tag{57}$$

The analysis of $SU(4')$ color gauge Yang-Mills theory is similar. Note that a fermions loop cannot interact with flavor and color gauge bosons in one triangle anomaly Feymann diagram at the same time. So, the $SU(4)_L \times SU(4)_R \times SU(4')$ Pati-Salam model is anomaly free.

7. MONOPOLE AND THE TOPOLOGY OF SPACETIME

As an example, we choose $SU(4)_L \times SU(4)_R$ flavor gauge bosons to analyze the problem of monopole. We can combine the $SU(4)_L \times SU(4)_R$ minimal coupling, Yang-Mills, and tolopogical terms of flavor gauge bosons as follows related to monopole:

$$\mathcal{L}_{\text{topology}} = -g\bar{\Psi}\gamma^\mu\Psi W_\mu - \frac{g^2\xi}{2}F^{\mu\nu}F_{\mu\nu} - \frac{\eta g^2\xi}{2}\widetilde{F}^{\mu\nu}F_{\mu\nu}, \tag{58}$$

where

$$\widetilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{59}$$

are electro-magnetic dual gauge strength tensors of $F_{\mu\nu}$. The Euler-Lagrangian equation of W_μ for the Lagrangian (58) is

$$2g\xi\partial_\mu F^{\mu\nu} + 2\eta g\xi\partial_\mu\widetilde{F}^{\mu\nu} = J^\nu, \quad J^\nu = \bar{\Psi}\gamma^\nu\Psi = J^\nu_e + J^\nu_m. \tag{60}$$

We decompose equation (60) as follows:

$$\partial_\mu F^{\mu\nu} = \frac{1}{2g\xi}J^\nu_e, \tag{61}$$
$$\partial_\mu\widetilde{F}^{\mu\nu} = \frac{1}{2\eta g\xi}J^\nu_m, \tag{62}$$

where J^ν_e is electro current and J^ν_m is monopole current. The fundamental thing in quantum field theory is action S

$$S = -\int_M \omega\left(\frac{\eta g^2\xi}{2}\widetilde{F}^{\mu\nu}F_{\mu\nu}\right), \tag{63}$$

where ω is volume form and M is the base manifold of spacetime.Note that the monopole-related topological term in (58) is the second Chern class, and the action S about the topological term only relies on the topological structure of the manifold M and is proportional with the second Chern number C_2

$$S \propto C_2, \quad C_2 \in \mathbb{Z}. \tag{64}$$

We can easily calculate the second Chern number C_2 with M equals topologies S^4 and $S^1 \times S^3$:

$$C_2 = 2, \quad M = S^4, \tag{65}$$

$$C_2 = 0, \quad M = S^1 \times S^3. \tag{66}$$

This means that, for base manifold M with topology S^4, the monopole about flavor gauge bosons W_μ, there are monopole currents; for base manifold M with topology $S^1 \times S^3$, the monopole currents are depressed. The analysis in this section could apply equally to $SU(4')$ color gauge bosons V_μ, and $SU(5)$ GUT also.

8. CONCLUSIONS AND DISCUSSION

Based on the gauge group $SU(4)_L \times SU(4)_R \times SU(4')$ of the Pati-Salam model, a representation of fermions is suggested in this paper. The boson-fermion-antifermion vertices brought by the $SU(4)_L \times SU(4)_R$ chiral flavor and the $SU(4')$ color gauge group were discussed. The electric charge of each particle was consistently defined, and a pair of possible CKM and PMNS matrix formulations were illustrated. An effective total Lagrangian of the model was given.

The experimental data restricts the masses of particles $X^{\pm C}$, Y^1, Y^1_*, Y^2, and Y^2_* were superheavy. How the masses be generated for these particles requires further discussion.

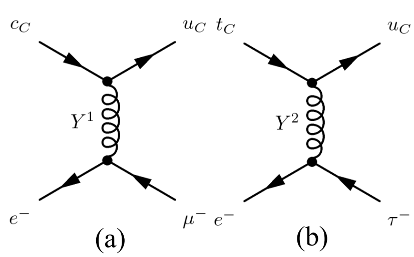

FIGURE 8: Examples of nonzero tree level amplitudes of the beyond SM FCNCs transported by neutral gauge bosons Y^1 and Y^2.

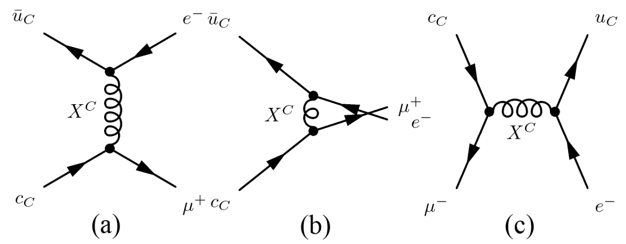

FIGURE 9: Nonzero tree-level amplitudes of semileptonic vertices. The e and μ are electron and mu leptons, and the u_C and c_C are u and c quarks with color C. These processes were still not being observed in the experiments, indicating that the masses of X bosons must have been superheavy. The u, e, and c, μ can be replaced with t, τ or d', ν' according to Lagrangian (24).

Appendix A. $SU(4)$ GROUP

Generators of the $SU(4)$ group are as follows:

$$
\begin{aligned}
&T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
&T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
&T^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
&T^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T^8 = \frac{\sqrt{3}}{6}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
&T^9 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad T^{10} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \\
&T^{11} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad T^{12} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \\
&T^{13} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad T^{14} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \\
&T^{15} = \frac{\sqrt{6}}{12}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.
\end{aligned}
\tag{A.1}
$$

Appendix B. THE POSSIBILITY OF THE CHIRAL SYMMETRY BREAKING OF FLAVOR GAUGE INTERACTION

$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ such that

$$\gamma^{5\dagger} = \gamma^5. \tag{B.2}$$

The gamma matrices satisfy

$$\begin{aligned} \frac{1-\gamma^{5\dagger}}{2}\frac{1+\gamma^5}{2} = 0, \quad \frac{1+\gamma^{\dagger}}{2}\frac{1-\gamma^5}{2} = 0, \\ \frac{1-\gamma^{5\dagger}}{2}\gamma^0\gamma^\mu\frac{1+\gamma^5}{2} = 0, \quad \frac{1+\gamma^{5\dagger}}{2}\gamma^0\gamma^\mu\frac{1-\gamma^5}{2} = 0, \end{aligned} \tag{B.3}$$

such that Lagrangian $-g\bar{\Psi}\gamma^\mu\Psi W_\mu$ can be decomposed into two chiral components in any Dirac matrix representation, i.e.,

$$-g\mathrm{Tr}\left[\bar{\Psi}\gamma^\mu\Psi W_\mu\right] = -g\mathrm{Tr}\left[\bar{\Psi}_L\gamma^\mu\Psi_L W_\mu^a T^a + \bar{\Psi}_R\gamma^\mu\Psi_R W_\mu^a T^a\right]. \tag{B.4}$$

Appendix C. CROSS SECTIONS

The cross section in Figure 8(a) can be represented as follows:

$$\begin{aligned} |M|^2 = \frac{16g^2}{\left(t - m_{\Upsilon^1}^2\right)^2} &\left(\left(s - m_e^2 - m_\mu^2\right)\left(s - m_u^2 - m_c^2\right)\right. \\ &+ \left(u - m_e^2 - m_\mu^2\right)\left(u - m_u^2 - m_c^2\right) \\ &+ 8m_e m_\mu m_u m_c + 2m_e m_\mu\left(t - m_u^2 - m_c^2\right) \\ &\left.+ 2m_u m_c\left(t - m_e^2 - m_\mu^2\right)\right). \end{aligned} \tag{C.5}$$

The cross section in Figure 8(b) replaces m_{Υ^1}, m_μ, and m_c with m_{Υ^2}, m_τ, and m_t, respectively.

The cross sections in Figures 9(a) and 9(b) are

$$\begin{aligned} |M_t|^2 = \frac{288f^4}{(t - m_X^2)^2} &\left(\left(s - m_\mu^2 - m_c^2\right)\left(s - m_e^2 - m_u^2\right)\right. \\ &+ \left(u - m_\mu^2 - m_c^2\right)\left(u - m_e^2 - m_u^2\right) \\ &+ 8m_e m_\mu m_u m_c + 2m_e m_\mu\left(t - m_\mu^2 - m_c^2\right) \\ &\left.+ 2m_\mu m_c\left(t - m_e^2 - m_u^2\right)\right), \end{aligned} \tag{C.6}$$

$$\begin{aligned} |M_u|^2 = \frac{288f^4}{(u - m_X^2)^2} &\left(\left(s - m_\mu^2 - m_c^2\right)\left(s - m_e^2 - m_u^2\right)\right. \\ &+ \left(t - m_\mu^2 - m_c^2\right)\left(t - m_e^2 - m_u^2\right) \\ &+ 8m_e m_\mu m_u m_c + 2m_e m_\mu\left(t - m_\mu^2 - m_c^2\right) \\ &\left.+ 2m_\mu m_c\left(t - m_e^2 - m_u^2\right)\right), \end{aligned} \tag{C.7}$$

where m_X, m_e, m_μ, m_u, and m_c are the masses of the $X^{\pm C}$ bosons, e, μ leptons, and u and c quarks. The s, t, and u are defined as follows:

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2, \tag{C.8}$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2, \tag{C.9}$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2. \tag{C.10}$$

The cross section of Figure 9(c) is given as follows:

$$\begin{aligned} |M_s|^2 = \frac{288f^4}{(s - m_X^2)^2} &\left(\left(t - m_\mu^2 - m_c^2\right)\left(t - m_e^2 - m_u^2\right)\right. \\ &+ \left(u - m_\mu^2 - m_c^2\right)\left(u - m_e^2 - m_u^2\right) \\ &+ 8m_e m_\mu m_u m_c + 2m_e m_\mu\left(s - m_\mu^2 - m_c^2\right) \\ &\left.+ 2m_\mu m_c\left(s - m_e^2 - m_u^2\right)\right). \end{aligned} \tag{C.11}$$

CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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