Entanglement Entropy Distributions of a Muon Decay

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Abstract

Divergences that occur in density matrices of decay and scattering processes are shown to be regularized by tracing and unitarity or the optical theorem. These divergences are regularized by the lifetime of the decaying particle or the total scattering cross section. Also, these regularizations are shown to give the expected helicities of final particles. As an illustration, the density matrix is derived for the weak decay of a polarized muon at rest, $\mu^- \to \nu_\mu(e^-\bar{\nu}_e)$, with Lorentz invariant density matrix entries and unitarity upheld at tree level. The electron's von Neumann entanglement entropy distributions are derived with respect to both the electron's emission angle and energy. The angular entropy distribution peaks for an electron emitted backward with respect to the muon's polarization given a minimum volume regularization larger than the cube of the muon's Compton wavelength. The kinematic entropy distribution is maximal at half the muon's rest mass energy. These results are similar to the electron's angular and kinematic decay rate distributions. Both the density matrix and entanglement entropy can be cast in terms of either ratios of areas or volumes.

Keywords: von Neumann entropy distribution, muon decay, density matrix, unitarity, optical theorem, divergences
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1. INTRODUCTION

Recently, the literature has applied the tools of quantum field theory to evaluate quantum information science (QIS) metrics for scalar and electromagnetic scattering [1, 2, 3, 4, 5, 6, 7, 8]. QIS metrics include the density matrix, von Neumann entanglement entropy, mutual information, etc. The density matrix carries information of the particles in terms of degrees of freedom such as momenta and polarizations. The entanglement entropy and mutual information are the degree of maximum knowledge and correlation between degrees of freedom, respectively. Reference [9] provides a review of QIS for particle physicists.

Reference [1] considers a scalar Φ^4-interaction and a time-dependent interaction. References [2, 3, 4, 5, 6, 7] consider an electromagnetic interaction, $e^-e^+ \to \mu^-\mu^+$. The Lorentz invariance of entanglement entropy is studied in [2]. Entropies and mutual information are calculated in both the helicity and spin basis in [3]. Reference [4] evaluates two scattering processes being correlated through an entanglement of initial particles. The paper [5] evaluates common QED processes such as Bhabha and Møller scattering. References [6, 7] consider a witness particle or spectator. During a scattering of two particles, one of these two is entangled with another particle (witness) that does not participate in the interaction. The scattering alters the reduced density matrix of the witness. This means the information of the witness particle is altered by the scattering event despite not participating directly in the interaction. However, the latter is not true when including unitarity [8]. Reference [8] also evaluates QIS metrics for Compton scattering wherein the scattering photon is entangled with a witness photon. In spite of not interacting directly, the electron and witness photon acquire a nonzero mutual information, i.e., become entangled, after the Compton scattering.

Since past works [1, 2, 3, 4, 5, 6, 7, 8] investigate scattering processes, QIS metrics are evaluated herein for a weak decay $\mu^- \to \nu_\mu(\bar{\nu}_e e^-)$. When deriving the density matrix, this work is distinguished from [1, 2, 3, 4, 5, 6, 7] by upholding unitarity up to tree level, i.e., using the optical theorem. For this reason, both the normalization of the density matrix and the total von Neumann entanglement entropy must be unchanged by the interaction. Only by keeping unitarity does the density matrix of final particles have a term representing the probability for *no* decay or *no* scattering to occur. Also, unlike [1, 2, 3, 4, 5, 6, 7, 8], this work regularizes common divergences that occur in the density matrices of decay and scattering processes. These divergences have been an obstacle in obtaining finite QIS metrics [1, 2, 3, 4, 5, 6, 7, 8]. Although the density matrix is derived for a decay process, the same techniques are readily applicable to scattering processes. A presentation of these methods is severely lacking in the literature.

In Section 2, the reduced density matrix of a daughter electron in the decay of a polarized muon is derived. The degrees of freedom are momentum and helicity. A common divergence appears in all terms and is regularized by tracing and the optical theorem. Furthermore, the expected helicity of the electron provides confirmation of the regularization. In Section 3, Lorentz invariant density matrix elements are extracted from the electron's reduced density matrix of momenta. These elements allow for calculating the von Neumann entanglement entropy distributions and a Lorentz invariant total entropy. The resulting angular and kinematic *entropy* distributions for the electron are peaked similarly to the corresponding *decay rate* distributions. An additive volume divergence occurs in the total Lorentz invariant entropy. Section 4 gives the regularized density matrix for the scattering process $e^-e^+ \to \sum_x x\bar{x}$.

2. DENSITY MATRIX OF MUON DECAY

Suppose a muon is prepared in a pure state $|i\rangle = |\mathbf{p}, \uparrow\rangle$ with momentum $\mathbf{p}$ and spin up along the $+z$-axis (see Figure 1). Its initial density matrix is $\rho^i = |i\rangle\langle i|$. The S-matrix or unitary operator gives the final state $|f\rangle = S|i\rangle$ where $S = 1 + i\mathcal{T}$ and $\mathcal{T}$ is the transition operator. The final state has a Fock space of 1 or 3 particles since the muon either does not decay or decays into $\nu_\mu(\bar{\nu}_e e^-)$. A free 1-particle and 3-particle Hamiltonian have bases that span the final state, implying a direct sum of final Hilbert spaces $\mathcal{H}_\mu \oplus \mathcal{H}_{\nu_\mu} \otimes \mathcal{H}_{\bar{\nu}_e} \otimes \mathcal{H}_e$. The three-particle state is written as $|\mathbf{l}_1, s_1; \mathbf{l}_2, s_2; \mathbf{l}_3, s_3\rangle = |\mathbf{l}_1, s_1\rangle \otimes |\mathbf{l}_2, s_2\rangle \otimes$

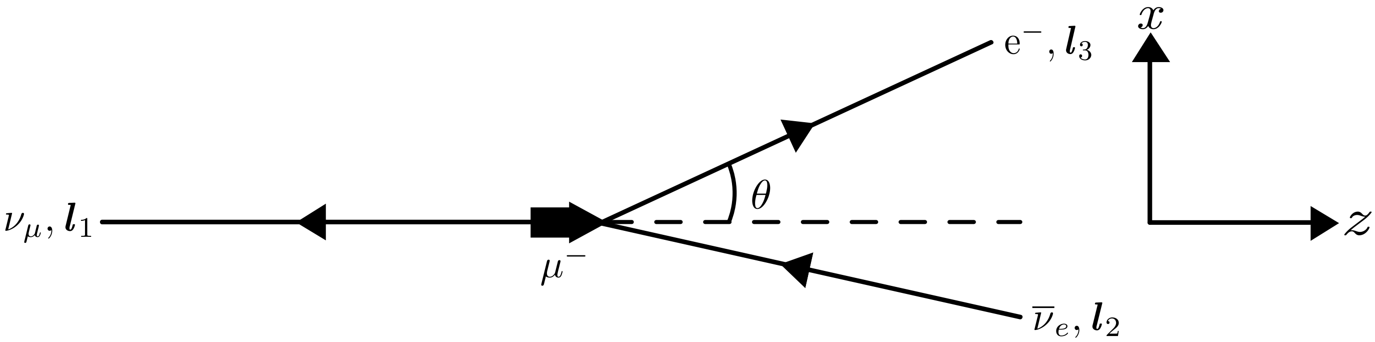


FIGURE 1: A polarized muon at rest decays into three particles: $\mu^- \to \nu_\mu(\bar{\nu}_e e^-)$.

$|\mathbf{l}_3, s_3\rangle$, where $(\mathbf{l}_i, s_i)$ are the momentum-polarization pairs for the three particles. The inner product of a state is $\langle \mathbf{p}, r|\mathbf{q}, s\rangle = 2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta_{r,s}$. Defining $Q_{\mathbf{l}s} \equiv \sum_s \int \frac{d^3\mathbf{l}}{(2\pi)^3 2E_{\mathbf{l}}}$, the single and three particle projection operators are written as

$$\begin{aligned} I_{\text{1-particle}} &= Q_{\mathbf{l}s}|\mathbf{l}, s\rangle\langle \mathbf{l}, s|, \\ I_{\text{3-particles}} &= \prod_{i=1}^{3} Q_{\mathbf{l}_i s_i} |\mathbf{l}_i, s_i\rangle \langle \mathbf{l}_i, s_i| \,. \end{aligned} \tag{1}$$

These particle projection operators are placed next to $i\mathcal{T}$ in the final state as follows.

$$\begin{aligned} |f\rangle &= S|i\rangle = (1+i\mathcal{T})|i\rangle \\ &= |i\rangle + \left(I_{\text{1-particle}} + I_{\text{3-particles}}\right) i\mathcal{T}|i\rangle. \end{aligned} \tag{2}$$

This gives a final density matrix $\rho^f = |f\rangle\langle f|$ or

$$\begin{aligned} \rho^f &= |i\rangle\langle i| \\ &+ \left(Q_{\mathbf{k}r}Q_{\mathbf{l}s}\langle \mathbf{l}, s|i\mathcal{T}|i\rangle\langle i| \left(-i\mathcal{T}^\dagger\right)|\mathbf{k}, r\rangle\right)|\mathbf{l}, s\rangle\langle \mathbf{k}, r| \\ &+ \prod_{m,n=1}^{3} Q_{\mathbf{k}_m r_m}Q_{\mathbf{l}_n s_n}\langle \mathbf{k}_m, r_m| i\mathcal{T}|i\rangle\langle i| \left(-i\mathcal{T}^\dagger\right)|\mathbf{l}_n, s_n\rangle \\ &* |\mathbf{k}_m, r_m\rangle\langle \mathbf{l}_n, s_n| \\ &+ \left(Q_{\mathbf{k}r}\langle \mathbf{k}, r|i\mathcal{T}|i\rangle|\mathbf{k}, r\rangle\langle i| + h.c.\right) \\ &+ \text{four other coherence terms.} \end{aligned} \tag{3}$$

Apart from the matrix positions, e.g., $|i\rangle\langle i|$, $|\mathbf{l}, s\rangle\langle \mathbf{k}, r|$, etc., the final density matrix, ρ^f, has Lorentz invariant entries. After tracing over all single particle states, the reduced density matrix for $\nu_\mu(\bar{\nu}_e e^-)$ is

$$\begin{aligned} \rho^f_{\nu_\mu\bar{\nu}_e e} &= \text{tr}_\mu\left(\rho^f\right) \\ &= 2E_{\mathbf{p}}V\left(1-T\Gamma\right) \\ &\quad + \prod_{m,n=1}^{3} Q_{\mathbf{k}_m r_m}Q_{\mathbf{l}_n s_n}\langle \mathbf{k}_m, r_m| i\mathcal{T}|i\rangle\langle i| \left(-i\mathcal{T}^\dagger\right)|\mathbf{l}_n, s_n\rangle \\ &\quad * |\mathbf{k}_m, r_m\rangle\langle \mathbf{l}_n, s_n| \,. \end{aligned} \tag{4}$$

Γ in the first term above is the muon's decay width. It occurs from the second and fifth lines in equation (3) by tracing and using unitarity or the optical theorem [10]. $V = (2\pi)^3\delta^3(0)$ and $T = 2\pi\delta(0)$ are the unregularized volume and time, respectively. Whether calculating density matrices of scattering or decay processes, these unregularized factors are a common occurrence [8].

The plus sign in equation (4) should be interpreted as a direct sum of the one-particle and three-particle states. The factor $(1-T\Gamma)$ in the first term can be interpreted as the probability for the muon *not* to decay, which must be zero. This implies that T is the inverse of the muon's total decay width Γ. Confirmation of T's regularization is seen below when calculating the e^-'s helicity and the trace of the density matrix. In short, setting the first term in equation (4) to zero gives the conditional density matrix for a polarized muon decay. Regardless of whether $(1-T\Gamma)$ is assumed to be zero, $\rho^f_{\nu_\mu\bar{\nu}_e e}$'s normalization is $\langle i|i\rangle = 2E_{\mathbf{p}}V$ after tracing over the three remaining particles. Hence, the normalization is unaffected by the decay. Notice if $I_{\text{1-particle}}$ is not included above equation (3), then unitarity or the constancy of the normalization is lost.

Tracing over the neutrinos, the normalized electronic reduced density matrix is

$$\begin{aligned} \rho^f_e &= \text{tr}_{\nu_\mu\bar{\nu}_e}\left(\rho^f_{\nu_\mu\bar{\nu}_e e}\right) \\ &= \sum_{t_3}\frac{T}{2m_\mu}\left(\prod_{m=1}^{3} Q_{\mathbf{l}_m s_m}\right) M^{\mathbf{l}_1,s_1;\mathbf{l}_2,s_2;\mathbf{l}_3,s_3}_{\mathbf{p},\uparrow}\left(M^{\mathbf{l}_1,s_1;\mathbf{l}_2,s_2;\mathbf{l}_3,t_3}_{\mathbf{p},\uparrow}\right)^\dagger \\ &\quad * (2\pi)^4\delta^{(4)}\left(p-\sum_{i=1}^{3} l_i\right)\left(\frac{|\mathbf{l}_3, s_3\rangle\langle \mathbf{l}_3, t_3|}{2EV}\right), \end{aligned} \tag{5}$$

where m_μ is the muon's mass, $E = l_3$ is the massless e^-'s energy, $M^{\mathbf{l}_1,s_1;\mathbf{l}_2,s_2;\mathbf{l}_3,s_3}_{\mathbf{p},\uparrow}$ is the Feynman amplitude for the μ^- decay, and the transition amplitude is related to the Feynman amplitude by

$$\langle \mathbf{l}_1, s_1; \mathbf{l}_2, s_2; \mathbf{l}_3, s_3| i\mathcal{T}|i\rangle = iM^{\mathbf{l}_1,s_1;\mathbf{l}_2,s_2;\mathbf{l}_3,s_3}_{\mathbf{p},\uparrow}(2\pi)^4\delta^{(4)}\left(p-\sum_{i=1}^{3} l_i\right). \tag{6}$$

The above electronic reduced density matrix lacks purity since $\rho^f_e \neq (\rho^f_e)^2$. Its trace is

$$\text{tr}\left(\rho^f_e\right) = T\Gamma = 1. \tag{7}$$

Just as ρ^f above, ρ^f_e's matrix elements, i.e., the coefficients of $\frac{|\mathbf{l}_3, s_3\rangle\langle \mathbf{l}_3, t_3|}{2EV}$, are Lorentz invariant. This follows from the Lorentz invariant factors: $T/2m_\mu = \frac{VT}{2m_\mu V}$, $Q_{\mathbf{l}_m,s_m}$, the Feynman amplitude, and $(2\pi)^4\delta^4(\cdot)$.

Assume in equation (5), the electron's polarization is its helicity, λ. In the massless limit for the electron, its helicity is $\lambda = -1$ and the ρ_e^f's polarization-coherence terms are zero. The helicity-reduced density matrix follows by tracing equation (5) over $\mathbf{l}_3$, giving

$$\rho_\lambda^f = \mathrm{tr}_{\mathbf{l}_3}\left(\rho_e^f\right) = T\Gamma|\lambda = -1\rangle\langle\lambda = -1|. \tag{8}$$

The latter correctly gives the electron's helicity,

$$\langle\sigma_z\rangle = \mathrm{tr}\left(\sigma_z\rho_\lambda^f\right) = -T\Gamma = -1, \tag{9}$$

where σ_z is Pauli's matrix. Conversely, the electron's helicity requires $T = \frac{1}{\Gamma}$. As another example, consider the pion decay $\pi^- \to e^-\bar{\nu}_e, \mu^-\bar{\nu}_\mu$. Using the above techniques, the expected antineutrino helicity is

$$T\sum_{x=e^-,\mu^-}\lambda_{\bar{\nu}_x}\Gamma_{\pi^-\to x\bar{\nu}_x} = 1 \tag{10}$$

since the antineutrinos' helicities are $+1$. Therefore, again, T must be the inverse of the total decay width.

Tracing equation (5) over the electronic polarizations gives a diagonal reduced density matrix of the electron's momenta.

$$\begin{aligned}\rho_{\mathbf{l}_3}^f &= \mathrm{tr}\left(\rho_e^f\right)\\ &= T\int\frac{d^3\mathbf{l}_3}{(2\pi)^3 2E}f(E,\theta)\frac{|\mathbf{l}_3\rangle\langle\mathbf{l}_3|}{2EV}, \quad \text{where}\\ f(E,\theta) &= \frac{G_F^2 m_\mu}{3\pi}E\left(3m_\mu - 4E + \cos\theta\left(m_\mu - 4E\right)\right),\end{aligned} \tag{11}$$

where $G_F = 1.1663788\times10^{-5}\,\mathrm{GeV}^{-2}$ is the Fermi coupling constant [11], $m_\mu = .1056583755\,\mathrm{GeV}$ is the muon's mass, $T = 2.1969811\times10^{-6}$ s is the muon's measured lifetime [12], $E = l_3$ is the massless electron's energy, and θ is the electron's scattering angle with respect to the muon's polarization [10].

The trace of any final density matrix being one requires upholding unitarity. This follows from $\mathrm{tr}(\rho) = \mathrm{tr}(|f\rangle\langle f|) = \langle f|f\rangle = \langle i|S^\dagger S|i\rangle = 1$ for a normalized initial state $|i\rangle$ and the unitary S-matrix. The trace of equation (11) is 1.0044 due to the calculation being at tree level. Hence, equation (11) approximately upholds unitarity since radiative corrections and other final particle states, e.g., photon emission or pair production, were not included. The decay modes of photon emission and pair production are negligible at branching fractions of 10^{-7} and 10^{-5} [12], respectively. Notice the integrand inside the above integral is the decay rate distribution $\frac{d^3\Gamma}{d^3\mathbf{l}_3} = \frac{f(E,\theta)}{(2\pi)^3 2E}$. Hence, adding these corrections would make the trace identically one, assuming there is no theoretical physics beyond the standard model.

3. VON NEUMANN ENTANGLEMENT ENTROPY DISTRIBUTIONS

Having the reduced density matrices for the electron in equations (5), (8), and (11), other quantum information metrics can be calculated. Since ρ_λ^f is pure, only the momentum contributes to the massless electron's von Neumann entanglement entropy,

$$\begin{aligned}S_e^{EE} &= -\mathrm{tr}\,\rho_e^f\log\rho_e^f = -\mathrm{tr}\,\rho_\lambda^f\log\rho_\lambda^f - \mathrm{tr}\,\rho_{\mathbf{l}_3}^f\log\rho_{\mathbf{l}_3}^f\\ &= -\mathrm{tr}\,\rho_{\mathbf{l}_3}^f\log\rho_{\mathbf{l}_3}^f\\ &\equiv S_{\mathbf{l}_3}^{EE}.\end{aligned} \tag{12}$$

The mutual information of the electron's momentum and helicity is

$$\begin{aligned}I &= S_{\mathbf{l}_3}^{EE} + S_\lambda^{EE} - S_e^{EE}\\ &= 0.\end{aligned} \tag{13}$$

This shows momentum and helicity are not correlated for a massless particle as expected for weak interactions.

The trace operator in the von Neumann entropy (S_e^{EE}) above follows from a continuous limit of the counting of states and recalling that $(2\pi\hbar)^3$ is the volume of phase space per state:

$$\sum_n \to \frac{V}{(2\pi\hbar)^3}\int d^3\mathbf{l}_3 = \delta^3(0)\int d^3\mathbf{l}_3. \tag{14}$$

In the last equality, the quantity $V = (2\pi\hbar)^3\delta^3(0)$ in nonnatural units was used. Equation (14) is also implied by tracing the identity operator for 1-particle states, giving the total number of states. By extracting $\delta^3(0)\int d^3\mathbf{l}_3$ from equation (11) and including $\hbar$ and c, the Lorentz invariant matrix elements of $\rho_{\mathbf{l}_3}^f$ are

$$\frac{Tf(E,\theta)\hbar^2c^3}{2EV} \tag{15}$$

or a ratio of the ostensible partial volume, $\varrho(E,\theta) \equiv \frac{Tf(E,\theta)\hbar^2c^3}{2E}$, to the total accessible volume, V. The cube root of the maximum partial volume is on the order of the muon's Compton wavelength, i.e., $\varrho_{\max}^{1/3} \sim \frac{h}{m_\mu c} \sim 10^{-14}$ m. With the partial volume, the electron's entanglement entropy distributions with respect to momentum and emission angle are calculated below and found to be similar to decay rate distributions. For the calculations below, assume log base e or units of *nats*. Alternatively, ϱ/V can be cast as a ratio of areas, i.e., $a(E,\theta)/A$ with $A \equiv \frac{V}{cT}$ and $a(E,\theta) \equiv f(E,\theta)(\hbar c)^2/(2E)$. Using equation (14), the density matrix in equation (11) can be rewritten in terms of ratios of areas or volumes.

$$\begin{aligned}\rho_{\mathbf{l}_3}^f &= \left(\frac{V}{(2\pi\hbar)^3}\int d^3\mathbf{l}_3\right)\frac{a(E,\theta)}{A}\frac{|\mathbf{l}_3\rangle\langle\mathbf{l}_3|}{2EV}\\ &= \left(\frac{V}{(2\pi\hbar)^3}\int d^3\mathbf{l}_3\right)\frac{\varrho(E,\theta)}{V}\frac{|\mathbf{l}_3\rangle\langle\mathbf{l}_3|}{2EV}.\end{aligned} \tag{16}$$

Using equation (15), the daughter electron's von Neumann entanglement entropy for the polarized muon decay is

$$\begin{aligned}S_e^{EE} &= -\mathrm{tr}\left(\rho_{\mathbf{l}_3}^f\log\rho_{\mathbf{l}_3}^f\right)\\ &= -\frac{V}{(2\pi\hbar)^3}\int d^3\mathbf{l}_3\left(\frac{\varrho(E,\theta)}{V}\log\left(\frac{\varrho(E,\theta)}{V}\right)\right).\end{aligned} \tag{17}$$

Hence, S_e^{EE} is manifestly Lorentz invariant when multiplying and dividing the trace operator by $2E$. The above equation looks similar to the Shannon entropy $\sum_i p_i\log p_i$.

After simplifying equation (17), the electron's entanglement entropy becomes

$$S_e^{EE} = -\frac{1}{(2\pi)^3\Gamma}\int d\Omega \int dE \frac{Ef(E,\theta)}{2}\log\left(\frac{f(E,\theta)(\hbar c)^3}{2\Gamma EV}\right), \tag{18}$$

where $\Gamma = \hbar/T$ is the muon's decay width. S_e^{EE} has three sources of entropy or missing information: positional uncertainty, momenta uncertainty, and quantum mechanical uncertainty. This is seen when expanding equation (18) as follows:

$$S_e^{EE} = \log V + \int d\Omega \int dE \frac{1}{\Gamma}\frac{d^2\Gamma}{d\Omega dE}\log\left(\frac{2\Gamma E}{f(E,\theta)c^3}\right) - 3\log\hbar. \tag{19}$$

In the above *information-theoretic* Sackur-Tetrode equation, the divergence in entropy, or $\log V$, is additive. The same occurs in the Rényi entropy. For an unregularized V, differences in entropy between different decay processes can yield finite results. Also, the mutual information between the muonic daughter particles' 3-momenta will be finite as the divergences in V would cancel. The argument of the logarithm in the above second term has dimensions of momentum cubed.

Differentiating equation (18) gives the electron's angular entropy distribution $\frac{dS_e^{EE}}{d\cos\theta}$ and kinematic entropy distribution $\frac{dS_e^{EE}}{dE}$, which are long equations and not written herein, see Figures 2(a) and 2(b). For regularizations above the muon's Compton wavelength cubed, i.e., $V \geq 1.34\varrho_{\max} = 1.1\times 10^{-41}\,\mathrm{m}^3$, the angular entropy distribution peaks at an electronic emission angle that is antiparallel with respect to the muonic polarization. For the latter regularization, the kinematic entropy distribution is nondecreasing and peaks at an energy of $E = m_\mu/2$ just as does the decay rate distribution $\frac{d\Gamma}{dE}$. As V increases, the entropy distributions approach their decay rate counterparts apart from an overall constant factor, i.e.,

$$\frac{d^2S_e^{EE}}{dE\, d\cos\theta} \to \frac{d^2\Gamma}{dE\, d\cos\theta}\frac{\log V}{\Gamma}. \tag{20}$$

The total electronic momentum entropy is $S_e^{EE} \geq .91\,\text{nats}$ for $V \geq 1.34\varrho_{\max}$. The electron's momentum entropy is nonzero since the neutrinos were traced out.

4. REGULARIZATION OF A SCATTERING PROCESS

For decay processes, evaluation of density matrices requires relating $T = 2\pi\delta(0)$ with the total decay width via $T = 1/\Gamma$. In other words, T is the muon's lifetime. On the other hand, for the density matrix of a scattering process, V/T should be related to the total accessible scattering cross section as shown below.

Consider the high energy annihilation process $e^-e^+ \to \sum_x x\bar{x}$ in the CoM frame where e^-, e^+ have particular helicities and $x, \bar{x}$ is a particle, antiparticle pair. Following the procedure of Section 2 and using the optical theorem, the final reduced density matrix of final particle x equals the direct sum

$$\rho^f_{e^-,\mu^-,\tau^-,\ldots} = \left(1 - \frac{\sum_f \sigma\left(e^-,e^+ \to f\right)}{\frac{V}{T\left|\boldsymbol{v}_{e^-,e^+}\right|}}\right) \oplus \frac{1}{\frac{V}{T\left|\boldsymbol{v}_{e^-,e^+}\right|}} \sum_{x=e^-,\mu^-,\tau^-,\ldots}\ \sum_{\lambda_x,\lambda_x'} \int d^3\mathbf{l}_x \frac{d^3\sigma^{\lambda_x,\lambda_x'}_{e^-,e^+\to x\bar{x}}}{d^3\mathbf{l}_x} * \frac{\left|\mathbf{l}_x,\lambda_x\right\rangle\left\langle \mathbf{l}_x,\lambda_x'\right|}{2E_{\mathbf{l}_x}V}. \tag{21}$$

$\mathbf{l}_x$ and λ_x above are the final particle x's momentum and helicity, respectively. $\boldsymbol{v}_{e^-,e^+}$ is the relative velocity of the initial particles. The sum over x is a direct sum, i.e., equation (21) is an array of square matrices in block diagonal form. The second term's differential cross section matrix elements are

$$\frac{d^3\sigma^{\lambda_x,\lambda_x'}_{e^-,e^+\to x\bar{x}}}{d^3\mathbf{l}_x} \equiv \frac{1}{2E_{e^-}2E_{e^+}\left|\boldsymbol{v}_{e^-,e^+}\right|}\frac{Q_{\mathbf{l}_{\bar{x}}\lambda_{\bar{x}}}}{(2\pi)^3E_{\mathbf{l}_x}} \times \left(M^{\mathbf{l}_x,\lambda_x;\mathbf{l}_{\bar{x}},\lambda_{\bar{x}}}_{e^-,e^+}\left(M^{\mathbf{l}_x,\lambda_x';\mathbf{l}_{\bar{x}},\lambda_{\bar{x}}}_{e^-,e^+}\right)^\dagger (2\pi)^4\delta^{(4)}\left(p - l_x - l_{\bar{x}}\right)\right). \tag{22}$$

Since the initial particles and final particles occupy different 2-particle vector spaces, the initial and final electrons are traced separately. The first term in equation (21) is from tracing over the initial particles e^- and e^+. For unitarity to hold exactly or $\mathrm{tr}(\rho) = 1$, all additional final particle states should be added to equation (21), e.g., including Bremsstrahlung or a final bosonic state ZH.

The above density matrix has Lorentz invariant entries, e.g., the ratio $\frac{\sum_f \sigma(a,b\to f)}{\frac{V}{T|\boldsymbol{v}_{a,b}|}}$ in the first term in equation (21) is manifestly Lorentz invariant [10] when rewritten as

$$\frac{2E_a2E_b\left|\boldsymbol{v}_{a,b}\right|\sum_f \sigma\left(a,b\to f\right)}{\left(\frac{2E_aV\,2E_bV}{VT}\right)}. \tag{23}$$

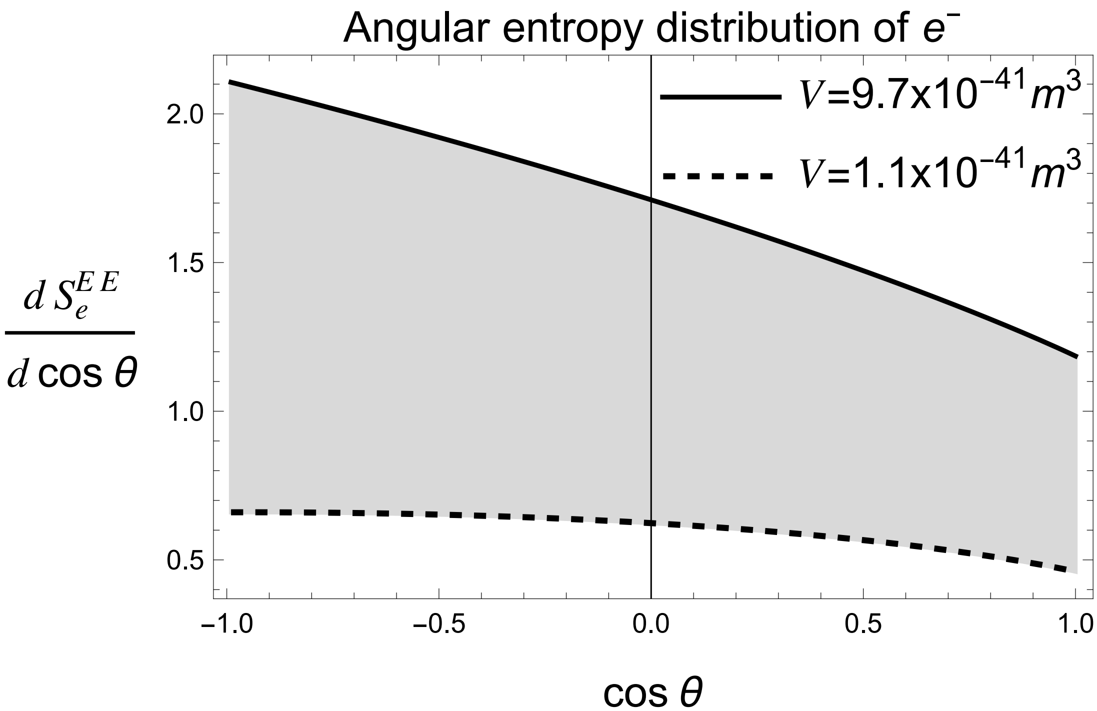
The first term in equation (21) is the probability for no scattering to occur. Setting this term to zero gives the conditional density matrix for e^-, e^+ scattering. This forces the *area* regularization for scattering to be

$$\frac{V}{T\left|\boldsymbol{v}_{e^-,e^+}\right|} = \sum_f \sigma\left(e^-,e^+\to f\right) \equiv \sigma_T. \tag{24}$$

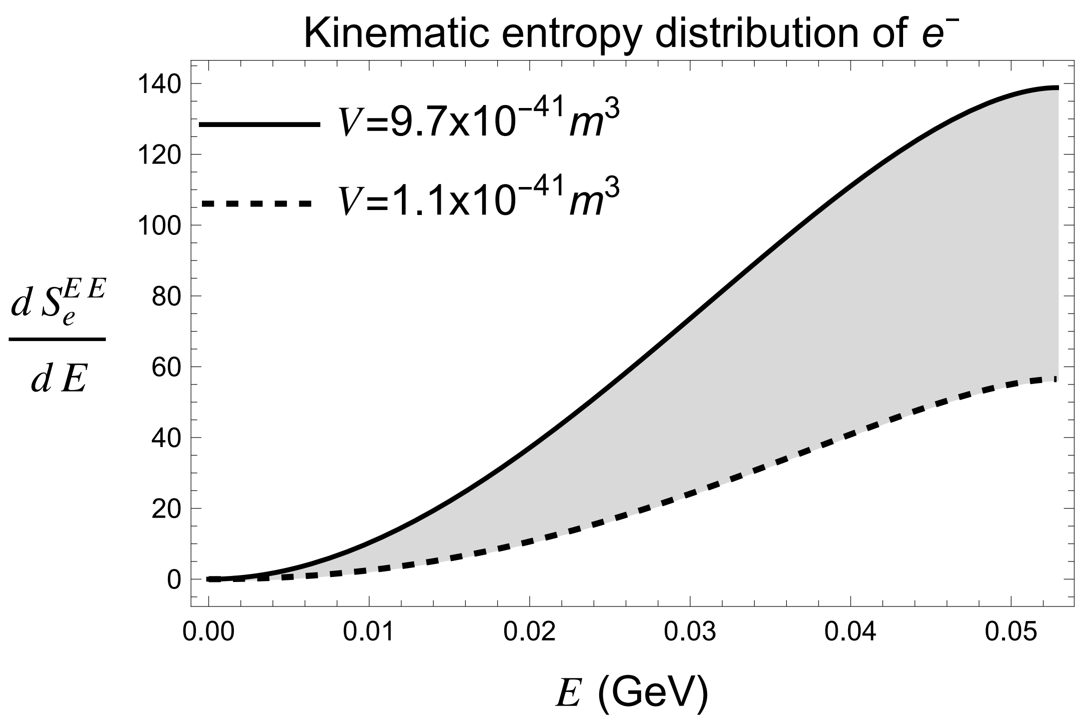
The latter equation can be interpreted as the interaction rate ($1/T$) divided by the luminosity ($v_{e^-,e^+}/V$) equals the total scattering cross-section (σ_T). With the latter regularization, the expected helicity of x from equations (21) and (24) is

$$\begin{aligned}\langle\sigma_z\rangle &= \mathrm{tr}\left(\sigma_z\sum_x\rho_x^f\right)\\ &= \sum_x\sum_{\lambda_x}\lambda_x\frac{\sigma^{\lambda_x,\lambda_x}_{e^-,e^+\to x\bar{x}}}{\sigma_T}.\end{aligned} \tag{25}$$

This is a weighted average of the helicities with each weight being the probability of a particular scattering process.



(a)



(b)

FIGURE 2: (a) θ is the emission angle of the electron with respect to the muon's polarization. V is the total volume accessible to the electron. For the angular distribution to be peaked at $\cos\theta = -1$, the lower bound of the regularized volume is $V \geq 1.1 \times 10^{-41}$ m^3, which is an order of magnitude larger than the muon's Compton wavelength cubed. As V grows, the distribution becomes proportional to its decay rate counterpart $\frac{d\Gamma}{d\cos\theta}$. The plot's sample regularization range is 1.1×10^{-41} m$^3 \leq V \leq 9.7 \times 10^{-41}$ m^3. (b) The massless e^-'s energy is $0 \leq E \leq \frac{m_\mu}{2}$, where m_μ is the μ^-'s rest mass.

5. DISCUSSION

The density matrix and von Neumann entanglement entropy have been derived for the decay of a polarized muon at rest, $\mu^- \to \nu_\mu(\bar{\nu}_e e^-)$. Since unitarity is upheld, the final and initial density matrices have the same normalization, which contrasts with works [1, 2, 3, 4, 5, 6, 7] that do not keep unitarity. The time divergence in the density matrix is resolved in Section 2 by tracing over μ^- and using the optical theorem. This time divergence is the muon's lifetime. Also, knowing the expected helicities of final particles suggests the same regularization. As a comparison, future work might consider an initial pure or mixed state of muonic polarizations with a helicity versus spin basis.

The algorithms presented herein for the muon decay are also applicable for evaluating quantum information metrics for any scattering process. For scattering processes, an *area* divergence occurs in the density matrix. In Section 4, the area divergence is identified with the total cross section. Both decay and scattering processes have the ratios of areas or volumes in their density matrix and von Neumann entanglement entropy.

When evaluating the Lorentz invariant von Neumann entanglement entropy for a muon decay, another unregularized volume, V, can appear. This volume is additive in the total entropy and reflects the positional uncertainty. Even if this accessible volume is not regularized, the mutual information between any two final state particles' 3-momenta will be finite. For a regularized volume exceeding the cube of the muon's Compton wavelength, the angular and kinematic entropy distributions for the daughter electron are found to be similar to the corresponding decay rate distributions. For the polarized muon decay, the reduced density matrix of a final state particle's momentum has three sources of entropy or missing information: positional uncertainty, momentum uncertainty, and quantum mechanical uncertainty. Further interactions should be investigated to find additional sources of uncertainties, perhaps by expanding the degrees of freedom beyond momenta and polarizations.

Future work should compare entanglement entropies, entropy distributions, and mutual information of daughter particles for different decay processes involving scalar, electromagnetic, weak, and strong interactions or even new physics. For example, the negative pion has mainly two decay modes, $\pi^- \to \mu^-\bar{\nu}_\mu$ or $e^-\bar{\nu}_e$. Helicity suppression can be reinterpreted as entropy suppression for the electronic mode.

CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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