Physical States and Correction Terms of the Supersymmetric $c = 1$ Model

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Abstract

In this article, we investigate the supersymmetric $c = 1$ model of superstring theory and demonstrate how the spectrum of states is expanded and new symmetries of the theory are generated by the existence of ghost cohomologies. As a result, we establish significant connections between two-dimensional supergravity and physical theories in higher dimensions. Additionally, we provide a comprehensive guide for constructing BRST-invariant and nontrivial vertex operators and carry out explicit computations to determine the correction terms needed to maintain the BRST invariance of the corresponding currents.

Keywords: supersymmetry, supergravity, ghost cohomology, string theory, BRST invariance
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1. INTRODUCTION

The formalism of ghost cohomologies is an approach used to study the nonperturbative dynamics of strings by exploring the important information contained in physical operators that are BRST- (Becchi-Rouet-Stora-Tyutin) invariant but not always manifestly gauge-invariant. This approach allows researchers to investigate the behavior of strings beyond the perturbative regime and gain a better understanding of the nonperturbative physics of gauge theories. In particular, it is used to study the properties of special vertex operators and their relation to nonperturbative strings. These vertex operators are not always manifestly BRST-invariant, meaning they do not commute with the supercurrent terms of the BRST charge. By adding *b*-*c* ghost-dependent terms to the expressions of these operators, researchers can restore their BRST invariance and better understand their properties. In string theory, the *b*-*c* ghosts are auxiliary fields that are introduced to make the worldsheet theory invariant under the BRST symmetry, which is a symmetry that plays a crucial role in the quantization of gauge theories. The *b* and *c* ghosts are worldsheet fields that satisfy a set of algebraic relations and can be thought of as anticommuting fields. The *b*-*c* ghost-dependent terms are terms that involve both the *b* and *c* ghosts and are added to the expressions of certain vertex operators in string theory. The addition of *b*-*c* ghost-dependent terms allows us to better understand the properties of these vertex operators and gain insights into the nonperturbative dynamics of strings. Overall, the formalism of ghost cohomologies provides a powerful tool for exploring the nonperturbative dynamics of strings and gaining insights into the physics of gauge theories.

In the first section, we review the concepts of BRST quantization and vertex operator formalism in the perturbative Ramond-Neveu-Schwarz (RNS) superstring theory [1, 2, 3, 4]. We also examine global spacetime symmetries and the generator of this symmetry, which is given by a special type of the worldsheet current violating the principle of ghost picture equivalence [5]. The perturbative RNS superstring theory is a type of string theory that describes the behavior of fundamental strings in terms of the motion of their worldsheet. The theory incorporates both bosonic and fermionic fields on the worldsheet and has supersymmetry, which allows for the cancellation of divergences that arise in the theory. In the RNS formalism, the worldsheet fields include the spacetime coordinates of the string, along with a set of fermionic fields known as spinors, which describe the fermionic degrees of freedom of the string. The theory has a set of symmetries, including Lorentz invariance and worldsheet reparametrization invariance, which constrain the dynamics of the theory. The perturbative RNS superstring theory is an important framework for studying the properties of strings and their interactions. It provides a foundation for the development of more advanced string theories, such as the superstring theories and *M*-theory, which incorporate additional symmetries and degrees of freedom.

We explore the properties of special vertex operators and their relation to nonperturbative strings, using examples of critical string theory and the supersymmetric $c = 1$ model [6, 7, 8]. The supersymmetric $c = 1$ model is a two-dimensional quantum field theory that describes the behavior of strings in a specific spacetime background. The theory incorporates both bosonic and fermionic fields on the worldsheet and has supersymmetry, which allows for the cancellation of divergences that arise in the theory. The $c = 1$ model has a central charge of one, which means that it has only one massless mode and is critical, meaning that it has no tachyons and is free of divergences. The theory has a set of symmetries, including Lorentz invariance and worldsheet reparametrization invariance, which constrain the dynamics of the theory. The supersymmetric $c = 1$ model is an important framework for studying the properties of strings and their interactions in a specific spacetime background. It has applications in areas such as condensed matter physics, where it can be used to model the behavior of certain two-dimensional materials. The study of this model has also led to insights into more general aspects of string theory and its connections to other areas of physics [9, 10, 11, 12].

We introduce the notion of ghost cohomologies and discuss their appearance based on the approach used in [13]. Section 2.3 explores the question of BRST invariance of vertex operators from the ghost cohomologies of positive and negative ghost numbers [14]. An interesting property of these vertex operators is that they are not manifestly BRST-invariant as they do not commute with the supercurrent terms of the BRST charge [15, 16, 17, 18]. The main importance of the general prescription that we provide here is that it allows restoring their BRST invariance by adding *b*-*c* ghost-dependent terms to the expressions of these operators and demonstrate how this strategy generally works on the example of the ghost-matter mixing five-form state of the critical NSR superstring theory in ten dimensions. More importantly, we provide an explicit calculation of the correction terms restoring the BRST invariance of the $T_{1,2}$ current. In the concluding section, we summarize our results, discuss some possible implications of our results, and suggest directions for future work.

2. THE RNS SUPERSTRING THEORY

String theory has been always considered a promising model that might be finally able to unify the fundamental interactions of nature, and it is also one of the best candidates for the construction of a consistent theory of quantum gravity. While the perturbative theory of strings seems to be already well explored, we still lack a complete and adequate string-theoretic formalism in the strongly coupled nonperturbative regimes. Strings were initially introduced in an attempt to find a solution for the problem of quark confinement as it is well-known that quarks exist in the bound state only as the interaction force between them grows with increasing distance [19]. Up until now, string theory has not been able to fully explain the quark confinement mechanism because it turned out to be hard to find an open string model whose partition function can exactly reproduce the expectation value of Wilson's loop in Quantum Chromodynamics (QCD) [19]. Nevertheless, several significant acheivements have been made in the string-theoretic approach to QCD in recent years [20, 21, 22]. AdS/CFT correspondence between string energies on the $AdS_5 \times S^5$ background with the anomalous dimensions of gauge-invariant operators in supersymmetric four-dimensional $SU(N)$ gauge theory, with several applications and extensions, has been successfully developed and studied [23, 24, 25, 26, 27, 28, 29, 30, 31]. Possible applications of string theory go far beyond QCD and the theory of strong interaction as well. This is why it is strongly believed that string theory appears to be a candidate for a model unifying gravity with the standard model.

String theory is a promising model that has the potential to unify fundamental interactions of nature and provide a consistent theory of quantum gravity. While the perturbative theory of strings has been extensively studied, a complete and appropriate string-theoretic formalism is still lacking in strongly coupled nonperturbative regimes. Originally, strings were proposed as a solution to the problem of quark confinement, which arises because quarks exist only in bound states as their interaction force increases with distance [19]. However, string theory has not yet fully explained the quark confinement mechanism due to the difficulty in finding an open string model that can accurately reproduce the expectation value of Wilson's loop in Quantum Chromodynamics (QCD). Despite this, significant progress has been made in the string-theoretic approach to QCD in recent years [20, 21, 22], particularly through the AdS/CFT correspondence between string energies on the $AdS_5 \times S^5$ background and the anomalous dimensions of gauge-invariant operators in supersymmetric four-dimensional $SU(N)$ gauge theory [23, 24, 25, 26, 27, 30, 31]. This correspondence has numerous applications and extensions, and the potential applications of string theory extend beyond QCD and the theory of strong interactions. Therefore, string theory is considered a promising candidate for a model that unifies gravity with the standard model [32, 33, 34].

First, we provide an overview of the fundamental concepts of string dynamics before extending them to the supersymmetric scenario. We use local coordinates, ξ_1 and ξ_2, to parameterize the worldsheet, and X_m $(m = 0, 1, \ldots, d-1)$ to denote the d-dimensional spacetime coordinates. The string action is defined as follows:

$$S_{\text{string}} = -\frac{1}{4\pi}\int d^2z\sqrt{\gamma}\gamma^{ab}\partial_a X^m \partial_b X^n\left(\xi_1, \xi_2\right)\eta_{mn}, \tag{1}$$

where $\gamma^{ab}(\xi_1, \xi_2)$ $(a, b = 1, 2)$ is the induced worldsheet metric and η^{mn} is the Minkowski metric [35]. In addition, the action is symmetric under the reparametrizations of the local worldsheet coordinates:

$$\begin{aligned} \xi_1 &\to f_1 = f_1\left(\xi_1, \xi_2\right), \\ \xi_2 &\to f_2 = f_2\left(\xi_1, \xi_2\right). \end{aligned} \tag{2}$$

The reparametrization symmetry (2) is a crucial local bosonic gauge symmetry in the framework of string theory. By making use of these gauge transformations, the worldsheet metric γ^{ab} can be transformed into the conformally flat form $\gamma^{ab} \to e^{\varphi}(\xi_1, \xi_2)\delta^{ab}$, where φ is a function of the worldsheet coordinates. Additionally, action (1) is invariant under Weyl rescalings of the metric, $\gamma^{ab} \to e^{\sigma}(\xi_1, \xi_2)\gamma^{ab}$. It is possible to eliminate the scale factor of e^{φ} in the worldsheet metric, which reduces action (1) to that with a flat metric. When subject to reparametrizations and Weyl rescalings, the integration measure undergoes the following transformations [35]:

$$D\left[\gamma^{ab}\right] \to D[\varphi]e^{-S_{\text{Liouville}}}D[b]D[c]e^{-S_{b\text{-}c}}. \tag{3}$$

The Faddeev-Popov determinant, which is similar to fixing the conformal gauge, arises from the action of the fermionic b-c reparametrization ghost fields $S_{bc} = \frac{1}{4\pi}\int d^2z(b\bar{\partial}c + \bar{b}\partial\bar{c})$ [1, 35, 36]. To make the calculation more convenient, the b and c ghost fields are bosonized using a single free bosonic field according to [37] with $b = e^{-\sigma}$ and $c = e^{\sigma}$. The Liouville field φ has an action given by

$$S_{\text{Liouville}} = \frac{D-26}{36\pi}\int d^zz\left(\partial\varphi\bar{\partial}\varphi + 2\mu_0 b e^{b\varphi}\right), \tag{4}$$

which reflects the anomaly of the functional integration measure $D[\varphi]$ under the Weyl rescaling. The cosmological constant constant is denoted by μ_0, and b is a constant determined by the Liouville field's background charge $Q = b + \frac{1}{b} = \sqrt{\frac{25-d}{3}}$ to make the total central charge of the system zero $c_X + c_{b\text{-}c} + c_{\varphi} = 0$ [1, 2, 35]. The central charge in conformal field theory (CFT) is determined from the two-point correlation functions of the stress-energy tensors of the appropriate fields [38, 39], such that

$$\langle T(z)T(w)\rangle = \frac{\frac{c}{2}}{(z-w)^4}, \tag{5}$$

where T is defined as T_{zz} and $T_{ab} = 2\pi(\gamma)^{\frac{1}{2}}\frac{\delta S}{\delta\gamma^{ab}}$.

By using the expressions for the stress-tensors of X, φ, and the ghost fields, the central charge can be calculated.

$$\begin{aligned} T_x &= -\frac{1}{2}\partial X_m \partial X^m, \\ T_{\varphi} &= -\frac{1}{2}(\partial\varphi)^2 + \frac{Q}{2}\partial^2\varphi, \\ T_{bc} &= \frac{1}{2}(\partial\sigma)^2 + \frac{3}{2}\partial^2\sigma \end{aligned} \tag{6}$$

together with the operator products:

$$\begin{aligned} X^m(z)X^n(w) &\sim -\eta^{mn}\ln(z-w), \\ \varphi(z)\varphi(w) &\sim -\ln(z-w), \\ \sigma(z)\sigma(w) &\sim \ln(z-w). \end{aligned} \tag{7}$$

It is possible to show that $c_X = d$, $c_\varphi = 1+3Q^2$, and $c_{b\text{-}c} = -26$. The full matter-ghost action is then given by

$$S = \frac{-1}{4\pi}\int d^2z\partial X_m\partial X^m + S_{\text{Liouville}} + S_{b\text{-}c}. \tag{8}$$

The RNS model incorporates two additional anticommuting Grassmann coordinates θ and $\bar\theta$ that extend the worldsheet coordinates $(z,\bar z)$. These Grassmann coordinates are defined in such a way that they satisfy certain conditions [40]:

$$\begin{aligned} &\theta^2 = \bar\theta^2 = 0, \quad \bar\theta\theta = -\theta\bar\theta, \\ &\int d\theta = 0, \quad \int d\theta\theta = 1. \end{aligned} \tag{9}$$

The worldsheet integration is replaced by the superspace integral $\int d^2z \to \int d^2z d^2\theta$ while concurrently replacing the derivatives in z and $\bar z$ by their covariant counterparts:

$$\partial_z \to D_z = \partial_\theta + \theta\partial_z, \quad \partial_{\bar z} \to D_{\bar z} = \partial_\theta + \bar\theta\partial_{\bar z}. \tag{10}$$

The expansions of the superfields are given by

$$\begin{aligned} X^m(z,\bar z,\theta,\bar\theta) &= X^m(z,\bar z) + \theta\psi^m(z,\bar z) + \bar\theta\bar\psi^m(z,\bar z) + \theta\bar\theta H^m(z,\bar z), \\ \varphi(z,\bar z,\theta,\bar\theta) &= \varphi(z,\bar z) + \theta\lambda(z,\bar z) + \bar\theta\bar\lambda(z,\bar z) + \theta\bar\theta F(z,\bar z), \\ C(z,\theta) &= c(z) + \theta\gamma(z), \\ \bar C(\bar z,\bar\theta) &= \bar c(\bar z) + \bar\theta\bar\gamma(\bar z), \\ B(z,\theta) &= \beta(z) + \theta b(z), \\ \bar B(\bar z,\bar\theta) &= \bar\beta(\bar z) + \bar\theta\bar b(\bar z). \end{aligned} \tag{11}$$

Integrating out θ and $\bar\theta$, it can be shown that the full ghost-matter action of the RNS superstring theory in the superconformal gauge is [41]

$$\begin{aligned} S_{\text{RNS}} &= -\frac{1}{4\pi}\int d^2z\left(\partial X_m\bar\partial X^m + \psi_m\bar\partial\psi^m + \bar\psi_m\partial\bar\psi^m\right) \\ &\quad + S_{\text{ghost}} + S_{\text{Liouville}}, \\ S_{\text{ghost}} &= \frac{1}{2\pi}\int d^2z(b\bar\partial c + \bar b\partial\bar c + \beta\bar\partial\gamma + \bar\beta\partial\bar\gamma), \\ S_{\text{Liouville}} &= \frac{d-10}{36\pi}\int d^2z\Big(\partial\varphi\bar\partial\bar\varphi + \lambda\bar\partial\lambda + \bar\lambda\partial\bar\lambda \\ &\qquad - F^2 + 2\mu_0 be^{b\varphi}(ib\lambda\bar\lambda - F)\Big), \\ Q &= b + \frac{1}{b} = \sqrt{\frac{9-d}{2}}. \end{aligned} \tag{12}$$

We bosonize the superconformal β and γ ghosts in terms of the pair of free $2d$ scalar bosons ϕ and χ. The bosonization relations are given by [37]

$$\begin{aligned} &\gamma = e^{\phi-\chi}, \quad \beta = e^{\chi-\phi}\partial_\chi, \\ &\langle\chi(z)\chi(w)\rangle = -\langle\phi(z)\phi(w)\rangle = \ln(z-w) \end{aligned} \tag{13}$$

and the full matter + ghost stress-energy tensor is

$$\begin{aligned} T_{\text{matter}} &= -\frac{1}{2}\partial X_m\partial X^m - \frac{1}{2}\partial\psi_m\psi^m, \\ T_{\text{ghost}} &= \frac{1}{2}(\partial\sigma)^2 + \frac{3}{2}\partial^2\sigma + \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}\partial^2\chi - \frac{1}{2}(\partial\phi)^2 - \partial^2\phi, \\ T_{\text{Liouville}} &= -\frac{1}{2}(\partial\varphi)^2 + \frac{Q}{2}\partial^2\varphi. \end{aligned} \tag{14}$$

Vertex Operators. In the string theory framework presented in this section, the oscillation modes are typically interpreted as fundamental particles, solitons, black holes, or D-branes. These entities are commonly described by vertex operators [1, 2, 42] with the following form:

$$V = P\left(\partial X^m, \partial^2X^m, \ldots, \psi^m, \partial\psi^m, \text{ghosts}\ldots\right)e^{ik_mX^m}. \tag{15}$$

Here, P is a polynomial in the fields and their derivatives, and k_m corresponds to the momentum. A vertex operator is deemed physical if it belongs to the BRST cohomology. The noteworthy characteristic of all gauge symmetries in such theories is that any local gauge symmetry of the theory automatically encompasses its invariance under another type of symmetry transformations, referred to as BRST symmetry [42, 43]. This implies that if an action of the theory is locally invariant under a certain local gauge symmetry, it would also be invariant under transformations of the same form as the gauge transformations, but with the local gauge parameter replaced by the corresponding Faddeev-Popov ghost of the opposite statistics. For the RNS model, the BRST charge expression is given precisely as

$$Q_{\text{BRST}} = \oint\frac{dz}{2i\pi}\left\{cT - bc\partial c - \frac{1}{2}\gamma\psi_m\partial X^m - \frac{1}{4}\gamma^2b\right\}. \tag{16}$$

To be considered physical vertex operators in the RNS model, they must be invariant under BRST transformations, meaning that they satisfy $Q_{\text{BRST}}, V = 0$. Operators that are BRST-exact and can be written as $V = Q_{\text{brst}}, W$ should be excluded. Open strings have an open line configuration, while closed strings have a loop configuration, with the simplest topology being a sphere. To be physical, these operators must be primary fields of dimension 1 for open strings and $(1,1)$ for closed strings. In conformal field theory (CFT), primary fields of dimension $(h,\bar h)$ are observables $\varphi^{h,\bar h}$ that transform under conformal transformations as $z \to f(z); \bar z \to f(\bar z)$ according to

$$\varphi^{h,\bar h}(z,\bar z) \to \left(\frac{df}{dz}\right)^h\left(\frac{d\bar f}{d\bar z}\right)^{\bar h}\left(f(z),\bar f(\bar z)\right). \tag{17}$$

Therefore, the operator product expansion (OPE) of the stress-energy tensor has a simple form:

$$T(z)\varphi^{h,\bar h}(w,\bar w) = \frac{h\varphi^{h,\bar h}(w,\bar w)}{(z-w)^2} + \frac{\partial\varphi^{h,\bar h}(w,\bar w)}{z-w} + O(z-w)^0. \tag{18}$$

3. DISCRETE STATES AND CURRENTS

3.1. Ghost Cohomologies in the RNS Model

Physical states in the RNS superstring theory are vertex operators that are both nontrivial and BRST-invariant. These operators can be defined up to transformations by the picture-changing operator $\Gamma = [Q_{\text{brst}}, \xi]$ and its inverse operator $\Gamma^{-1} =$

$c\partial\xi e^{-2\phi}$, where $\xi = e^{\chi}$ and ϕ, χ is the pair of the bosonized superconformal ghosts, which consists of the product of the BRST charge and the pair of bosonized superconformal ghosts. The inverse of the picture-changing operator increments or decrements the ghost number of the operator by 1. Thus, each string excitation can be described by an infinite set of physically equivalent operators.

$$\begin{aligned} :\Gamma V^{(n)} &:= V^{(n+1)} + \{Q_{\text{brst}}, \dots\}, \\ :\Gamma^{-1} V^{(n)} &:= V^{(n-1)} + \{Q_{\text{brst}}, \dots\}. \end{aligned} \tag{19}$$

The location of picture-changing operators inside correlation functions can vary, resulting in a full derivative in the supermoduli space. This ensures their picture invariance under appropriate moduli integration. However, global singularities arise if the correlation function contains vertex operators that diverge faster than $(z - z_n)^{-2}$, where z_n is the insertion point on the worldsheet [44]. If such an operator is present, the moduli integration of the full derivative term would result in a nonzero boundary contribution, and the correlation function would be picture-dependent. An example of such a vertex operator in the critical RNS superstring theory is the five-form state:

$$V_5(k) \equiv H_{m_1\cdots m_5}(k) \oint \frac{dz}{2i\pi} e^{-3\phi} \psi^{m_1} \cdots \psi^{m_5} e^{ikX}(z). \tag{20}$$

This operator exists at all the negative pictures below -3 but has no versions at pictures -2 or higher [5]. Since V_5 is annihilated by Γ, we have

$$\begin{aligned} 0 &\equiv \lim_{w\to z} \Gamma^{-1}(u)\Gamma(w)V_5(z) \\ &= V_5(z) + \lim_{w\to z} \{Q_{\text{brst}}, \Lambda(u,w)V_5(z)\}. \end{aligned} \tag{21}$$

Hence, V_5 is the BRST commutator: $V_5(z) = -\lim_{w\to z}\{Q_{\text{brst}}, \Lambda(u,w)V_5(z)\}$. It can be written as

$$\begin{aligned} &\lim_{w\to z} \{Q_{\text{brst}}, \Lambda(u,w)V_5(z)\} \\ &\quad = \lim_{w\to z} \left\{Q_{\text{brst}}, \Gamma^{-1}(u)(\xi(w) - \xi(u))V_5(z)\right\} \\ &\quad = \lim_{w\to z} \Gamma^{-1}(u)(\Gamma(w) - \Gamma(u))V_5(z) \end{aligned} \tag{22}$$

and the commutator is given by $V_5(z) = \{Q_{\text{brst}}, \xi\Gamma^{-1}(u)V_5(z)\}$. Since the derivatives of Γ are all BRST-trivial: $\partial^n\Gamma = \{Q_{\text{brst}}, \partial^n\xi\}$; $n = 1, 2, \dots$, and one can write

$$\Gamma(w) = \Gamma(z) + \left\{Q_{\text{brst}}, \sum_n \frac{(w-z)^n}{n!}\partial^n\xi\right\}. \tag{23}$$

The nonsingular OPE of Γ with U_5 is given by

$$\Gamma(z)U_5(w) \sim (z-w)^2 e^{-2\phi} \psi^{m_1} \cdots \psi^{m_4} \left(i(k\psi)\psi^{m_5} + \partial X_{m_5}\right) e^{ikX}. \tag{24}$$

Our goal here is to derive the correction terms restoring the BRST-invariance of W_5 and to demonstrate the nonzero correlator involving the V_5 and W_5 operators. We start with the BRST invariance restoration, and we consider the BRST charge given by

$$Q_{\text{brst}} = \oint \frac{dz}{2i\pi} \left(cT - bc\partial c - \frac{1}{2}\gamma\psi_m\partial X^m - \frac{1}{4}\gamma^2 b\right). \tag{25}$$

Using the operator $L(z) = -4ce^{2\chi-2\phi} \equiv: \xi\Gamma^{-1}$, we find that it satisfies $\{Q_{\text{brst}}, L\} = 1$. Next, consider a noninvariant operator V satisfying $\{Q_{\text{brst}}, V\} = W$, then W is BRST-exact, and the transformation $V \to V_{\text{inv}} = V - LW$ restores BRST-invariance. Applying the same scheme for W_5, we have

$$\begin{aligned} [Q_{\text{brst}}, W_5] = H_{m_1\cdots m_5}(k) &\oint \frac{dz}{2i\pi} e^{2\phi-\chi+ikX} R_1^{m_1\cdots m_5}(z) \\ &+ be^{3\phi-2\chi+ikX} R_1^{m_1\cdots m_5}(z), \end{aligned} \tag{26}$$

where

$$\begin{aligned} R_1^{m_1\cdots m_5}(z) = &-\frac{1}{2}\psi^{m_1}\cdots\psi^{m_5}(\psi\partial X) \\ &-\frac{1}{2}\psi^{[m_1\cdots m_4}\left(\partial^2 X^{m_5]} + \partial X^{m_5]}(\partial\phi - \partial\chi)\right) \\ &-\frac{i}{2}\psi^{m_1}\cdots\psi^{m_5}(k\psi)(\partial\phi - \partial\chi) + (k\partial\psi), \end{aligned} \tag{27}$$

$$R_1^{m_1\cdots m_5}(z) = -\frac{1}{4}(2\partial\phi - 2\partial\chi\partial\sigma)\psi^{m_1}\cdots\psi^{m_5}. \tag{28}$$

Evaluating the OPE of L, we obtain

$$\begin{aligned} &W_{5\text{inv}}(k,w) \\ &= H_{m_1\cdots m_5}(k)\left\{\oint \frac{dz}{2i\pi} e^{\phi}\psi^{m_1}\cdots\psi^{m_5}e^{ikX} - \frac{1}{2}\oint_w \frac{dz}{2i\pi}(z-w)^2 \right. \\ &\qquad \left. : L\partial_z^2\left(e^{2\phi-\chi}R_1^{m_1\cdots m_5} + e^{3\phi-2\chi}R_2^{m_1\cdots m_5}(z)\right): \right\} \\ &= H_{m_1\cdots m_5}(k)\left\{\oint \frac{dz}{2i\pi} e^{\phi}\psi^{m_1}\cdots\psi^{m_5}e^{ikX} \right. \\ &\qquad \left. - 2\oint_w \frac{dz}{2i\pi}(z-w)^2 ce^{\chi}R_1^{m_1\cdots m_5}(k,z)\right\}. \end{aligned} \tag{29}$$

There are three types of ghost cohomologies: positive ghost number cohomology, negative ghost number cohomology, and zero ghost cohomology. A formal definition of ghost cohomologies was given in [5, 13] and is summarized by

(1) The positive ghost number cohomology, denoted as H_N (where N is a positive integer), consists of physical vertex operators that exist at positive superconformal ghost pictures $n \geq N$ and are annihilated by the inverse picture-changing operator Γ^{-1} at picture N. This means that picture N is the minimal positive picture at which the operators $V \subset H_N$ can exist.

(2) The negative ghost number cohomology, denoted as H_{-N} (where N is a positive integer), consists of physical vertex operators that exist at negative superconformal pictures $n \leq N$ and are annihilated by the direct picture-changing at picture $-N$.

(3) By definition, zero ghost cohomology, denoted as H_0, consists of operators that exist in all pictures. Standard string perturbation theory involves elements of H_0. The standard string perturbation theory thus involves the elements of H_0. The picture -3 and picture $+1$ five-forms considered above are the elements of H_{-3} and H_1, respectively.

(4) There is a generic isomorphism between positive and negative ghost cohomologies: $H_{-N-2} \sim H_N$; $N \geq 1$. This means that to any element of H_{-N-2}, there corresponds an element from H_N obtained by replacing $e^{-(N+2)\phi}$ with $e^{N\phi}$ and then adding the appropriate b-c ghost terms in order to restore the BRST-invariance.

The example of the picture -3 and picture $+1$ five-forms discussed earlier belongs to H_{-3} and H_1, respectively. By understanding ghost cohomologies, and researchers in string theory can better understand the behavior of strings in different spacetime backgrounds and calculate various physical quantities such as scattering amplitudes and correlation functions.

3.2. *Extended Discrete States in $c = 1$ Supersymmetric Model*

Noncritical one-dimensional string theory is known to include discrete states with nonstandard *b-c* ghost numbers 0 and 2 in its physical state spectrum. If we consider the $c = 1$ model, which is supersymmetrized on the worldsheet and coupled to the super Liouville field, the action of the system on the worldsheet in conformal gauge is given by

$$\begin{gathered} S = S_{X-\psi} + S_L + S_{b\text{-}c} + S_{\beta-\gamma}, \\ S_{X-\psi} = \frac{1}{4\pi}\int d^2z \left\{\partial X\bar{\partial}X + \psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}\right\}, \\ S_L \\ = \frac{1}{4\pi}\int d^2z \left\{\partial\varphi\bar{\partial}\varphi + \lambda\bar{\partial}\lambda + \bar{\lambda}\partial\bar{\lambda} - F^2 + 2\mu_0 b e^{b\varphi}(ib\lambda\bar{\lambda} - F)\right\}, \\ S_{b\text{-}c} + S_{\beta-\gamma} = \frac{1}{4\pi}\int d^2z \left\{b\bar{\partial}c + \bar{b}\partial\bar{c} + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma}\right\}, \end{gathered} \tag{30}$$

where $Q \equiv b + b^{-1}$ is the background charge. The stress tensors of the matter and the ghost systems and the standard bosonization relations for the ghosts are given by

$$\begin{gathered} T_m = -\frac{1}{2}(\partial X)^2 - \frac{1}{2}\partial\psi\psi - \frac{1}{2}(\partial\varphi)^2 + \frac{Q}{2}\partial^2\varphi, \\ T_{gh} = \frac{1}{2}(\partial\sigma)^2 + \frac{3}{2}\partial^2\sigma + \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}\partial^2\chi - \frac{1}{2}(\partial\phi)^2 - \partial^2\phi, \\ c = e^{\sigma}, \quad b = e^{-\sigma}, \\ \gamma = e^{\phi-\chi}, \quad \beta = e^{\chi-\phi}\partial\chi. \end{gathered} \tag{31}$$

According to the prescription provided by [13], the $SU(2)$ algebra is generated by the currents:

$$\begin{aligned} T_{0,0} &= \oint \frac{dz}{2i\pi}\partial X, \\ T_{0,1} &= \oint \frac{dz}{2i\pi}e^{iX}\psi, \\ T_{0,-1} &= \oint \frac{dz}{2i\pi}e^{-iX}\psi. \end{aligned} \tag{32}$$

It is important to note the currents of the form

$$T_{-n,m} = \oint \frac{dz}{2i\pi}P_{-n,m}\left(\partial X, \partial^2 X, \dots, \psi, \partial\psi, \dots\right)e^{-n\phi+imX}, \quad |m| \le n-1 \tag{33}$$

The Virasoro primary states with negative ghost number $-n$, which are typically not BRST invariant, are usually annihilated by the picture-changing operator. The states $P_{-n,m}$ are polynomial expressions in ∂X, ψ, and their derivatives, with a conformal weight of $h = 1/2(n^2 - m^2) + n + 1$, ensuring that the integrals have a total dimension of 1. To begin, one starts with the Liouville-dressed tachyonic Virasoro primaries for a given n

$$\oint \frac{dz}{2i\pi}V_l = \oint \frac{dz}{2i\pi}e^{ilX+(l-1)\varphi}(l\psi - i(l-1)\varphi)$$

with integer l and acts on them with various combinations of the lowering T-operators. The obtained operators will be the multiplets of $SU(n)$, including the operators of BRST cohomologies with nontrivial ghost dependence.

3.2.1. *Ghost Dependent Discrete States*

We consider here three $SU(2)$ currents taken at different ghost pictures. The first example is the generator given by the worldsheet integral

$$T_{-3,2} = \oint \frac{dz}{2i\pi}e^{-3\phi+2iX}\psi(z). \tag{34}$$

This operator is annihilated by the picture-changing transformation. By taking the lowering operator $T_{0,-1} = \oint \frac{dz}{2i\pi}e^{-iX}\psi(z)$ of $SU(2)$, we obtain the following extra five generators in the ghost number -3 cohomology:

$$\begin{aligned} T_{-3,2} &= \oint \frac{dz}{2i\pi}e^{-3\phi+2iX}\psi(z), \\ T_{-3,1} &= \oint \frac{dz}{2i\pi}e^{-3\phi+iX}\left(\partial\psi\psi + \frac{1}{2}(\partial X)^2 + \frac{i}{2}\partial^2 X\right)(z), \\ T_{-3,-1} &= \oint \frac{dz}{2i\pi}e^{-3\phi-iX}\left(\partial\psi\psi + \frac{1}{2}(\partial X)^2 - \frac{i}{2}\partial^2 X\right)(z), \\ T_{-3,0} &= \oint \frac{dz}{2i\pi}e^{-3\phi}\left(\partial^2 X\psi - 2\partial X\partial\psi\right)(z), \\ T_{-3,-2} &= \oint \frac{dz}{2i\pi}e^{-3\phi-2iX}\psi(z). \end{aligned} \tag{35}$$

The following step involves demonstrating that when combined with the three standard $SU(2)$ generators $T_{0,0}$, $T_{0,1}$, and $T_{0,-1}$, the operators form eight generators of $SU(3)$, where $T_{0,0}$ and $T_{-3,0}$ create the Cartan subalgebra of $SU(3)$. The computation of the commutators of some of the generators can be complicated. However, it can be simplified by noting that the operators $T_{-n,m}$ of ghost cohomology $-n$ (where $n = 3, 4, \dots$) are equivalent to operators from the positive ghost number $n-2$ cohomologies, up to certain *b-c* ghost terms necessary for maintaining the BRST-invariance of the operators with positive superconformal ghost numbers. To simplify the process, it is convenient to redefine the operators

$$\begin{aligned} L &= \frac{i}{2}T_{0,0}, \\ H &= \frac{i}{3\sqrt{2}}T_{-3,0}, \\ G_+ &= \frac{1}{2\sqrt{2}}\left(\sqrt{2}T_{0,1} + T_{-3,1}\right), \\ G_- &= \frac{1}{2\sqrt{2}}\left(\sqrt{2}T_{0,1} - T_{-3,1}\right), \\ F_+ &= \frac{1}{2\sqrt{2}}\left(\sqrt{2}T_{0,-1} + T_{-3,-1}\right), \\ F_- &= \frac{1}{2\sqrt{2}}\left(\sqrt{2}T_{0,-1} - T_{-3,-1}\right), \\ G_3 &= \frac{1}{\sqrt{2}}T_{-3,2}, \\ F_3 &= \frac{1}{\sqrt{2}}T_{-3,-2}. \end{aligned} \tag{36}$$

Consequently, the commutators of the operators L and H are given by

$$\begin{gathered}[L,H]=0,\\ [L,G_+]=\frac{1}{2}G_+;\quad [L,G_-]=\frac{1}{2}G_-;\quad [L,F_+]=-\frac{1}{2}F_+;\\ [L,F_-]=-\frac{1}{2}F,\\ [L,G_3]=G_3;\quad [L,F_3]=-F_3,\\ [H,G_+]=-G_+;\quad [H,G_-]=G_-;\quad [H,F_+]=F_+;\\ [H,F_-]=-F_-,\\ [H,G_3]=[H,F_3]=0.\end{gathered} \tag{37}$$

In summary, it has been demonstrated that the operators L and H are in the Cartan subalgebra of $SU(3)$. Thus, the operators L, H, $F_\pm$, $G_\pm$, F_3, and G_3 define the Cartan-Weyl basis of $SU(3)$. Therefore, just like in the case of usual $SU(2)$ discrete states of two-dimensional supergravity, this operator is $V=\oint\frac{dz}{2i\pi}e^{ilX+(l-1)\varphi}(l\psi-i(l-1)\lambda)$. In particular, to include the currents of ghost numbers up to -4, one has to start with the generator $T_{-4,3}=\oint\frac{dz}{2i\pi}e^{-4\phi+3iX}\psi$. A similar procedure can be used to include the currents of higher ghost numbers, such as $T_{-4,3}$, which generates seven new currents $T_{-4,m}$, $|m|\le 3$ that are the BRST-invariant super Virasoro primaries. These currents can be obtained by repeatedly applying $T_{0,-1}$ to $T_{-4,3}$. The resulting operators are described in [13]:

$$\begin{aligned}
&T_{-4,\pm3}=\oint\frac{dz}{2i\pi}e^{-4\phi\pm3iX}\psi(z),\\
&T_{-4,2}\\
&=\oint\frac{dz}{2i\pi}e^{-4\phi+2iX}\left(\frac{1}{2}\partial^2\psi\psi-\frac{i}{6}\partial^3X+\frac{i}{6}(\partial X)^3-\frac{1}{2}\partial X\partial^2X\right),\\
&T_{-4,1}\\
&=\oint\frac{dz}{2i\pi}e^{-4\phi+iX}\left(\frac{1}{2}\psi\partial\psi\partial^2\psi+\frac{1}{24}P^{(4)}_{-iX}\psi\right.\\
&\qquad\left.-\frac{1}{4}P^{(2)}_{-iX}\partial^2\psi-\frac{1}{4}\left(P^{(2)}_{-iX}\right)^2\psi\right),\\
&T_{-4,-1}\\
&=\oint\frac{dz}{2i\pi}e^{-4\phi-iX}\left(\frac{1}{2}\psi\partial\psi\partial^2\psi+\frac{1}{24}P^{(4)}_{iX}\psi\right.\\
&\qquad\left.-\frac{1}{4}P^{(2)}_{iX}\partial^2\psi-\frac{1}{4}\left(P^{(2)}_{iX}\right)^2\psi\right),\\
&T_{-4,-2}\\
&=\oint\frac{dz}{2i\pi}e^{-4\phi-2iX}\left(\frac{1}{2}\partial^2\psi\psi+\frac{i}{6}\partial^3X-\frac{i}{6}(\partial X)^3-\frac{1}{2}\partial X\partial^2X\right),\\
&T_{-4,0}\\
&=\oint\frac{dz}{2i\pi}e^{-4\phi}\left\{2i\partial X\partial\psi\partial^2\psi+P^{(2)}_{-iX}\psi\partial^2\psi-\frac{2}{3}P^{(3)}_{-iX}\psi\partial\psi\right.\\
&\qquad-\frac{1}{6}P^{(3)}_{-iX}P^{(2)}_{-iX}+(\partial X)^2\psi\partial^2\psi+\frac{7i}{8}\partial XP^{(4)}_{-iX}\\
&\qquad-i(\partial X)^3\psi\partial\psi-\frac{i}{2}\partial XP^{(2)}_{-iX}\psi\partial\psi\\
&\qquad\left.+\frac{i}{4}\partial X\left(P^{(2)}_{-iX}\right)^2-\frac{1}{4}(\partial X)^2P^{(3)}_{-iX}\right\}.
\end{aligned}\tag{38}$$

Here, $P^{(n)}_{\pm iX}$; $n=2,3,4$ are the conformal weight-n polynomials in the derivatives of X defined as $P^{(n)}_{f(X(z))}=e^{-f(X(z))}\frac{\partial^n}{\partial z^n}e^{f(X(z))}$ for a given function $f(X)$. When one applies the lowering subalgebra of $SU(4)$ to the dressed tachyonic vertex multiple times, the set of ghost-dependent discrete states is extended, and they become the multiplets of $SU(4)$. It seems reasonable to assume that this process can be generalized to include generators of higher ghost numbers.

3.2.2. Nontriviality of the $T_{-n,m}$-Currents

The BRST charge of the one-dimensional NSR superstring theory is given by the usual worldsheet integral

$$Q_{\rm brst}=\oint\frac{dz}{2i\pi}\left\{cT-bc\partial c+\gamma G_{\rm matter}-\frac{1}{4}b\gamma^2\right\},\tag{39}$$

where $G_{\rm matter}$ is the full matter supercurrent. The BRST-invariant currents are given by $T_{n-2,m}=Z(:\Gamma^{2n-2}cS_{-n,m}:)$ [44] while $S_{-n,m}$ are the integrands of $T_{-n,m}$ and Z is the picture changing operators for the b-c ghost terms [37] which simply follow from the invariance of Γ and Z. If we can demonstrate that any operator creates nonzero correlations, we can prove its BRST nontriviality. The T-currents automatically demonstrate the BRST nontriviality of new discrete states in $SU(n)$ multiplets. There are two ways to express the W_n operator, which, when its commutator with the BRST charge is taken, may generate the T-currents

$$\begin{gathered}W_n=W_n^{(1)}+W_n^{(2)},\\ W_n^{(1)}=\sum_{k=1}^{n-1}\alpha_k\oint\frac{dz}{2i\pi}e^{-(n+1)\phi+i(n-1)X}\partial^{(k)}\xi\partial^{(n-k)}X,\\ W_n^{(2)}=\sum_{k,l=1,k\neq l}^{k,l=n,k+l\le 2n}\alpha_{kl}\oint\frac{dz}{2i\pi}e^{-(n+2)\phi+i(n-1)X}\psi\partial^{(k)}\xi\partial^{(2n-k-l)}c\end{gathered}\tag{40}$$

with α_k and α_{kl} being some coefficients and $\xi=e^X$. The operators $W_n^{(1)}$ and $W_n^{(2)}$ are the conformal one-dimensional operators satisfying the relations

$$\begin{gathered}\left[\oint\frac{dz}{2i\pi}\gamma^2\psi\partial X,W^{(1)}\right]\sim T_{-n,n+1},\\ \left[\oint\frac{dz}{2i\pi}\gamma^2 b,W_n^{(2)}\right]\sim T_{-n,n-1},\\ \left[\oint\frac{dz}{2i\pi}\gamma^2\psi\partial X,W^{(2)}\right]=\left[\oint\frac{dz}{2i\pi}\gamma^2 b,\ W_n^{(21)}\right]=0.\end{gathered}\tag{41}$$

Therefore, the T-currents are BRST-trivial if and only if there exists at least one combination of the coefficients α_k or α_{kl} such that

$$\left[\oint\frac{dz}{2i\pi}(cT-bc\partial c),W_n^{(1)}\right]=0,\tag{42}$$

$$\left[\oint\frac{dz}{2i\pi}(cT-bc\partial c),W_n^{(2)}\right]=0.\tag{43}$$

3.3. $SU(n)$ Multiplets and Structure Constants

We create sets of ghost-related specific states belonging to $SU(N)$ and determine their structure constants when N is equal to 3. The discrete states resulting from the lower operators of the current algebra lead to a variety of $SU(3)$ representations. We start with the decomposition of the current algebra [45]

$$SU(3)=N_+\oplus N_0\oplus N_-\tag{44}$$

with the operators L and H being in the Cartan subalgebra N_0, the subalgebra N_+ consisting of 3 operators $G_\pm$ and G_3 with the unit positive momentum and with 3 lowering operators $F_\pm$ and F_3 with the unit negative momentum being in N_-.

The discrete states forming the $SU(3)$ multiplets can be obtained by the various combinations of the N_- operators with the highest weight vector states. The highest weight vectors are given by the dressed tachyonic operators $\oint \frac{dz}{2i\pi}V_l(z) = \oint \frac{dz}{2i\pi}e^{ilX+(l-1)\varphi}(l\psi(z)-i(l-1)\lambda)$, where l is an integer. It is easy to see that all the V_l's with $l \geq 2$ are annihilated by N_+ since their OPE's with the integrands of $G_\pm$ and G_3 are nonsingular. Furthermore, all these tachyons have a weight of $\frac{1}{2}$ with respect to the L and N_0 operators. We need to examine how the hypercharge generator H of N_0 acts on these operators. Simple calculation gives

$$\begin{aligned}&\left[H, \oint \frac{dz}{2i\pi}V_l\right]\\ &= \left[\frac{i}{3\sqrt{2}}\frac{dz}{2i\pi}e^{-3\phi}\left(\partial^2X\psi - 2\partial X\partial\psi\right),\right.\\ &\qquad\left.\oint\frac{dw}{2i\pi}e^{ilX}(l\psi - i(l-1)\lambda)(w)\right]\\ &= \frac{i}{3\sqrt{2}}\frac{dw}{2i\pi}e^{-3\phi+ilX+(l-1)\varphi}\\ &\quad\times\left\{3il^2\partial\psi\psi - 3l\partial^2X + \frac{1}{2}il^2P^{(2)}_{-3\phi}\right.\\ &\qquad\left.+(6l\partial X - 3l(l-1)\psi\lambda)\partial\phi + 3l(l-1)\partial\psi\lambda\right\}.\end{aligned}\tag{45}$$

Performing the partial integration, we get

$$\begin{aligned}&\left[H, \oint \frac{dz}{2i\pi}V_l\right]\\ &= \frac{i}{3\sqrt{2}}\oint\frac{dz}{2i\pi}e^{-3\phi+ilX+(l-1)\varphi}\\ &\quad\times\left\{3il^2\partial\psi\psi - l(1+\frac{l^2}{2})\partial^2X + il^2(l-1)\partial^2\varphi\right.\\ &\qquad + l(l-1)(2\partial\psi\lambda - \psi\partial\lambda)\\ &\qquad + (il\partial X + (l-1)\partial\varphi)(2l\partial X - l(l-1)\psi\lambda)\\ &\qquad\left.+\frac{il^2}{2}(il\partial X + (l-1)\partial\varphi)^2\right\}.\end{aligned}\tag{46}$$

The expression on the right side of the equation is the dressed tachyon in picture -3, with the exception of a factor that is related to hypercharge. For the purposes of this analysis, we only need to examine the matter component of the picture-changing operator.

$$\Gamma =: \delta(\beta)G_{\text{matter}} := -\frac{i}{\sqrt{2}}e^{\phi}(\psi\partial X + \lambda\partial\varphi + \partial\lambda).\tag{47}$$

Applying the picture-changing operator gives

$$\begin{aligned}&:\Gamma :: \left[H, \oint\frac{dz}{2i\pi}V_l\right]:\\ &= -\frac{i}{\sqrt{2}}\frac{i}{3\sqrt{2}}\oint\frac{dz}{2i\pi}e^{-2\phi+ilX}\psi(z)\times(2-2l),\end{aligned}\tag{48}$$

i.e., the tachyon at the picture -2. We can also show that [13]

$$:\Gamma^3 : \left[H, \oint\frac{dz}{2i\pi}V_l(z)\right] = \frac{l(l-1)}{6}\oint\frac{dz}{2i\pi}V_l\tag{49}$$

and this proves that the tachyons with the integer momenta $l \geq 2$ are the highest weight vectors of $SU(3)$. Once we have identified the highest weight vectors, we can obtain the physical states spectrum, which consists of the $SU(3)$ multiplets, by straightforwardly constructing the corresponding vertex operators:

$$\oint\frac{dz}{2i\pi}V_{l;p_1,p_2,p_3} = F_+^{p_1}F_-^{p_2}F_3^{p_3}\oint\frac{dz}{2i\pi}V_l(z)\tag{50}$$

with all possible integer values of p_1, p_2, and p_3 such that $p_1 + p_2 + 2p_3 \leq 2l$.

4. CORRECTION TERMS OF THE $T_{1,2}$ CURRENT

At this stage, it is crucial to demonstrate that the new discrete states of the model's currents are BRST-invariant. This is achieved through correction terms associated with the current. Here, we perform a specific calculation of the correction terms that restore the BRST invariance for $T_{1,2}$, which is the current in ghost cohomology $+1$ and the dual current of the $T_{-3,2}$ current that exists in ghost cohomology -3. Using the prescription $V \to V_{\text{inv}} = V - LW$, with $W = \{Q_{\text{BRST}}, V\}$, we start by calculating

$$\begin{aligned}&\{Q_{\text{brst}}, T_{1,2}\}\\ &= \oint\frac{dz}{2i\pi}\left\{-\frac{1}{2}e^{\phi-\chi}\psi\partial X - \frac{1}{4}e^{2\phi-2\chi}b\right\}\oint\frac{dw}{2i\pi}e^{\phi+2iX}\\ &= -\frac{1}{2}\oint\frac{dw}{2i\pi}e^{2\phi-\chi+2iX}\left(\partial^2X + \partial X(\partial\phi\partial\chi)\right)\\ &\quad-\frac{1}{4}\oint\frac{dw}{2i\pi}e^{3\phi-2\chi+2iX}\psi P^{(1)}_{2\phi-2\chi\sigma},\end{aligned}\tag{51}$$

where $P^{(n)}_{(f(z))} = W^{(1)} + W^{(2)}$. As proved before, there exists an operator $L(z) = -4ce^{2\chi-2\phi}$ satisfying $\{Q_{\text{BRST}}, L\} = 1$. Making the transformation: $T^{\text{inv}}_{1,2} = T_{1,2} - LW$, it is straightforward to notice that $\{Q, T^{\text{inv}}_{1,2}\} = 0$. The process for calculating the correction terms goes as follows:

(i) Take : $L(z)W(w)$: and expand around the midpoint $\frac{z+w}{2}$ up to the second order $(z-w)^2$ terms.

(ii) Calculate the limit $\lim_{z\to w}\{\frac{1}{2}\oint\frac{dz}{2i\pi}(z-u)^2\partial_z^2L(z)U(z)\}$.

(iii) Integrate the obtained answer by parts to get an answer of the form:

$$\begin{aligned}&a\oint\frac{dz}{2i\pi}e^{\phi+2iX}\psi + \oint\frac{dz}{2i\pi}e^{\phi+2iX}\psi(z-u)\text{ (extra terms)}\\ &\qquad+\oint\frac{dz}{2i\pi}e^{\phi+2iX}\psi(z-u)^2\text{ (extra terms)}.\end{aligned}\tag{52}$$

These terms are the correction terms of $T_{1,2}$ which are explicitly calculated from:

$$\begin{aligned}&: L(z)W^{(1)}(w):\\ &= \lim_{z\to w}2\oint e^{2\chi-2\phi+\sigma}(z)e^{2\phi-\chi+2iX}\\ &\qquad\times\left(\partial^2X + \partial X(\partial\phi - \partial\chi)\right)(w)\frac{dz}{2i\pi}\\ &= 2\oint(z-u)^2ce^{\chi+2iX}\left(\partial^2X + \partial X(\partial\phi - \partial\chi)\right)\frac{dz}{2i\pi},\end{aligned}\tag{53}$$

$$: L(z)W^{(2)}(w) :$$
$$= \lim_{z\to w}\oint e^{2\chi-2\phi+\sigma}(z)e^{3\phi-2\chi+2iX}\psi(2\partial\phi-2\partial\chi-\partial\sigma)(w)\frac{dz}{2i\pi}. \tag{54}$$

Performing the expansion around $\frac{z+w}{2}$ and keeping the terms up to the second order, we obtain the following.

$z-w$ terms:

$$\lim_{z\to w}\oint\frac{dz}{2i\pi}e^{\phi+2iX}(z-w)\times\Big\{\psi(2\partial\phi-2\partial\chi-\partial\sigma)+\psi\left(\partial\chi-\partial\phi+\frac{\partial\sigma}{2}-\frac{3}{2}\partial\phi+\partial\chi-iX+\frac{\partial\sigma}{2}\right)-\frac{\partial\psi}{2}\Big\}. \tag{55}$$

Further integration implies that

$$\lim_{z\to w}-\oint\frac{(z-u)^2}{2}\partial_z\left\{e^{\phi+2iX}\left(\psi\left(-\frac{\partial\phi}{2}-iX\right)-\frac{\partial\psi}{2}\right)\right\}\frac{dz}{2i\pi}$$
$$=\oint\frac{dz}{2i\pi}\frac{(z-u)^2}{2}e^{\phi+2iX}\times\Big\{\psi\left(\frac{\partial^2\phi}{2}+i\partial^2X-(\partial\phi+2i\partial X)\left(-\frac{\partial\phi}{2}-i\partial X\right)\right)-\partial\psi(-\partial\phi-2iX)+\frac{\partial^2\psi}{2}\Big\}. \tag{56}$$

$(z-w)^2$ terms:
The $(z-w)^2$ correction terms are obtained from

$$\lim_{z\to w}\oint\frac{dz}{2i\pi}e^{\phi+2iX}(z-w)^2\times\Big\{-\frac{1}{2}\left(2\partial^2\phi-2\partial^2\chi-\partial^2\sigma\right)\psi-\frac{\partial\psi}{2}(2\partial\phi-2\partial\chi-\partial\sigma)+\psi(2\partial\phi-2\partial\chi-\partial\sigma)\times\left(\partial\chi-\partial\phi+\frac{\partial\sigma}{2}-\frac{3}{2}\partial\phi+\partial\chi-iX+\frac{\partial\sigma}{2}\right)-\frac{\partial\psi}{2}$$
$$+\frac{1}{8}\left\{\left(2\partial^2\chi-2\partial^2\phi+\partial^2\sigma\right)+(2\partial\chi-2\partial\phi+\partial\sigma)^2\right\}\psi+\frac{1}{8}\left\{\left(3\partial^2\phi-2\partial^2\chi+2i\partial^2X-\partial^2\sigma\right)+(3\partial\phi-2\partial\chi+2i\partial X-\partial\sigma)^2\right\}\psi$$
$$+\frac{1}{4}(3\partial\phi-2\partial\chi+2i\partial X-\partial\sigma)\partial\psi+\frac{1}{8}\partial^2\psi-\frac{1}{4}(2\partial\chi-2\partial\phi+\partial\sigma)(3\partial\phi-2\partial\chi+2i\partial X-\partial\sigma)\psi-\frac{1}{4}(2\partial\chi-2\partial\phi+\partial\sigma)\partial\psi\Big\}. \tag{57}$$

Collecting terms and integrating over a closed loop, then taking the term involving $\partial^2\psi$ and integrating by parts twice, and then involving the $\partial\psi$ term and integrating by parts once, we obtain the following.

$\partial^2\psi$ term:

$$\frac{3}{8}\oint\frac{dz}{2i\pi}(z-u)^2e^{\phi+2iX}\partial^2\psi=\frac{3}{8}\oint\frac{dz}{2i\pi}e^{\phi+2iX}\times\Big\{2+4(z-u)(\partial\phi+2i\partial X)+(z-u)^2\left(\partial^2\phi+2i\partial^2X+(\partial\phi+2i\partial X)^2\right)\Big\}\psi. \tag{58}$$

$\partial\psi$ term:

$$\frac{1}{2}\oint\frac{dz}{2i\pi}(z-u)^2e^{\phi+2iX}\partial\psi\left\{\frac{5}{2}\partial\phi+3i\partial X\right\}=-\frac{1}{2}\oint\frac{dz}{2i\pi}e^{\phi+2iX}\times\Big\{\psi(z-u)^2\left(\frac{5}{2}\partial^2\phi+3i\partial^2X+\left(\frac{5}{2}\partial\phi+3i\partial X\right)(\partial\phi+2i\partial x)\right)+2(z-u)\left(\frac{5}{2}\partial\phi+3i\partial X\right)\Big\}. \tag{59}$$

ψ term:

$$\frac{1}{2}\oint\frac{dz}{2i\pi}(z-u)^2e^{\phi+2iX}\times\Big\{\psi\Big\{-\partial^2\phi+\frac{9}{4}\partial^2\chi+\frac{3}{2}i\partial^2X+\partial^2\sigma+\frac{1}{2}(\partial\phi+2i\partial X)^2+(2\partial\phi-2\partial\chi-\partial\sigma)\left(\frac{7}{4}\partial\chi-\frac{9}{2}\partial\phi+\frac{7}{4}\partial\sigma-2i\partial X\right)+(3\partial\phi-2\partial\chi+2i\partial X-\partial\sigma)\times\left(-\frac{3}{2}\partial\chi+\frac{5}{4}\partial\phi-\frac{3}{4}\partial\sigma+\frac{i}{2}\partial X\right)\Big\}\Big\}. \tag{60}$$

Further computation from the terms just obtained shows that the correction terms of the current implied by equation (51) finally simplify into

$$\frac{3}{8}\oint\frac{dz}{2i\pi}e^{\phi+2iX}\psi+\oint\frac{dz}{2i\pi}(z-u)e^{\phi+2iX}\psi(-\partial\phi)+\frac{1}{2}\oint\frac{dz}{2i\pi}(z-u)^2e^{\phi+2iX}\psi\times\Big\{-\frac{11}{4}\partial^2\phi+\frac{9}{4}\partial^2\chi+\partial^2\sigma-(\partial\phi+i\partial X)(\partial\phi+2i\partial X)+(2\partial\phi-2\partial\chi-\partial\sigma)\left(\frac{7}{4}\partial\chi-\frac{9}{2}\partial\phi+\frac{7}{4}\partial\sigma-2i\partial X\right)+(3\partial\phi-2\partial\chi+2i\partial X-\partial\sigma)\times\left(-\frac{3}{2}\partial\chi+\frac{5}{4}\partial\phi-\frac{3}{4}\partial\sigma+\frac{i}{2}\partial X\right)\Big\}. \tag{61}$$

In this passage, we have described a new result that has been obtained by fully determining the correction terms of the $T_{1,2}$ current, which is a current that carries the ghost cohomology $+1$ and is dual to the $T_{-3,2}$ current at ghost cohomology -3. The importance of this result lies in its ability to restore the BRST invariance of the current, which is a crucial property for any physically meaningful operator in the context of the supersymmetric $c=1$ model being investigated.

Furthermore, we note that this algorithm can be applied to all current cohomologies, providing a general prescription for finding the additional terms that restore the BRST invariance of any noninvariant current carrying ghost cohomology at a given picture. This approach is therefore significant in that it proves the existence of new discrete physical states in the supersymmetric $c = 1$ model and can be used as a general method for investigating other similar models.

5. CONCLUSION

This article discussed the enhancement of the current algebra of spacetime generators in noncritical RNS superstring theories due to the appearance of new physical ghost-dependent generators from the first nontrivial ghost cohomology. The current algebra of spacetime generators has significant applications in string theory, enabling the calculation of various physical quantities such as scattering amplitudes and correlation functions. Additionally, the current algebra has connections to other areas of physics such as conformal field theory and quantum field theory, making it an important area of study.

The author demonstrated that the $SU(2) \sim SL(3, R)$ algebra of currents is isomorphic to volume-preserving diffeomorphisms in three dimensions. This suggests that there are holographic relations between two-dimensional supergravity and field-theoretic degrees of freedom in three dimensions. The conjecture is that by including currents from the cohomologies of ghost numbers up to N, the current algebra can be extended to $SU(N + 2)$, corresponding to volume-preserving diffeomorphisms in $d = N + 2$ dimensions. Thus, each new cohomology corresponds to opening up a theory to a new hidden spacetime dimension in the $c = 1$ supersymmetric model. The authors proved this fact for $N = 1$ and conjecture for higher N values. They also provide an explicit construction of the correction terms for the currents with higher ghost cohomologies, which gives an isomorphism between positive and negative ghost cohomologies. The $c > 1$ case is not discussed in this work as it is more complicated.

CONFLICTS OF INTEREST

The author declares that there are no conflicts of interest regarding the publication of this paper.

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