Lepton Masses and Mixing in Modular A_5 Symmetry

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Abstract

A model based on the modular group A_5 is considered to account for lepton masses and mixing. We consider multimoduli scenario, in which charged leptons and neutrinos are assigned to different moduli. Various models are considered depending on different assignments of modular weights and the mechanism for producing the light neutrino masses.

Keywords: neutrino, modular flavor symmetries, discrete symmetries
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1. INTRODUCTION

The finite modular groups Γ_N have been considered to account for the flavor problems [1, 2]. In these groups, the coupling constants can transform nontrivially; in addition, extra symmetries under modular weights are considered. Some of Γ_N are isomorphic to finite permutation groups. For instance, $\Gamma_2 \cong S_3$ [3, 4, 5, 6], $\Gamma_3 \cong A_4$ [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22], $\Gamma_4 \cong S_4$ [23, 24, 25, 26, 27, 28, 29, 30] and $\Gamma_5 \cong A_5$ [31, 32, 33]. Some approaches have been studied to consider two moduli with different VEVs of fixed points for charged leptons and neutrinos [34, 35]. On the other hand, multiple modular groups have been broken effectively into a single modular group with multimoduli discussed in [36]. In this work, we adopt the scenario of two moduli with modular group $\Gamma_5 \sim A_5$. This would allow us to assign different moduli to neutrino and charged lepton sectors.

2. MODULAR FORMS OF LEVEL 5

Group A_5 has 60 elements and five irreducible representations, namely, **1**, **3**, $3'$, **4**, and **5**, and is generated by two elements S and T satisfying the following conditions:

$$S^2 = T^5 = (ST)^3 = \mathbf{1}. \tag{1}$$

The modular form of level 5 has the following form:

$$f_i(\gamma(\tau)) = (c\tau + d)^{2k} \rho_{ij}(\gamma) f_j(\tau), \quad \gamma \in \Gamma(5). \tag{2}$$

The modular forms of weight 2 have been calculated in [33, 31]:

$$Y_3 = \begin{pmatrix} e_1(\tau) \\ e_2(\tau) \\ e_3(\tau) \end{pmatrix}, \quad Y_{3'} = \begin{pmatrix} e'_1(\tau) \\ e'_2(\tau) \\ e'_3(\tau) \end{pmatrix}, \quad Y_5 = \begin{pmatrix} \tilde{e}_1(\tau) \\ \tilde{e}_2(\tau) \\ \tilde{e}_3(\tau) \\ \tilde{e}_4(\tau) \\ \tilde{e}_5(\tau) \end{pmatrix}, \tag{3}$$

where the elements of modular forms are written in terms of the Dedekind eta-function $\eta(\tau)$:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i \tau}, \tag{4}$$

and the Klein form

$$\mathfrak{k}_{(r_1,r_2)}(\tau) = q_z^{(r_1-1)/2} (1 - q_z) \prod_{n=1}^{\infty} (1 - q^n q_z) \left(1 - q^n q_z^{-1}\right) (1 - q^n)^{-2}, \tag{5}$$

where $q_z = e^{2\pi i z}$.

In this paper, we will use the basis where the triplet irreducible representation of A_5 generators is given by

$$S = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\phi & 1/\phi \\ -\sqrt{2} & 1/\phi & -\phi \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2i\pi}{5}} & 0 \\ 0 & 0 & e^{-\frac{2i\pi}{5}} \end{pmatrix}, \tag{6}$$

where $\phi = \frac{1+\sqrt{5}}{2}$. The fixed points in the fundamental domain are

$$\tau_1 = e^{\frac{i2\pi}{3}} = \frac{-1}{2} + i\frac{\sqrt{3}}{2}, \quad \tau_2 = i. \tag{7}$$

There are other fixed points but they are equivalent to the above points by modular transformation. For instant, $\tau_{1'} = \frac{1}{2} + i\frac{\sqrt{3}}{2} = T\tau_1$, $\tau_{2'} = -0.5 + 0.5i = ST\tau_2$, $\tau_{2''} = 0.5 + 0.5i = TST\tau_2$.

3. LEPTON MASSES AND MIXING

We consider the following scenarios that depend on different assignments of flavors under modular A_5 and modular weights. In each of these models, the charged lepton mass and mixing matrices are not changed.

3.1. Charged Lepton Sector

The assignments under A_5 and modular weights for the lepton and Higgs fields in this model are shown in Table 1. For the charged leptons, according to the modular invariance condition, for the modular forms of weight 2, the modular weights in the charged lepton sector must satisfy the following conditions:

$$k_L + k_{H_d} + k_E = 2. \tag{8}$$

Fields	E^c	L	N^c	H_u	H_d	
A_5	3	3	3	1	1	
	1	1	0	0	0	model 1
k_I	0	2	0	0	0	model 2
	2	0	1	0	0	model 3

TABLE 1: Assignment of flavors under A_5 and the modular weight k_I.

The invariant superpotential under the modular A_5 group is

$$w_e = g_1 \left(E^c H_d L\right)_3 Y_3 \left(\tau_e\right) + g_2 \left(E^c H_d L\right)_5 Y_5 \left(\tau_e\right). \quad (9)$$

The charged lepton mass matrix in this case is

$$m_e = v_d \begin{pmatrix} 2g_2\tilde{e}_1 & g_1e_3 - \sqrt{3}g_2\tilde{e}_5 & -g_1e_2 - \sqrt{3}g_2\tilde{e}_2 \\ -g_1e_3 - \sqrt{3}g_2\tilde{e}_5 & \sqrt{6}g_2\tilde{e}_4 & g_1e_1 - g_2\tilde{e}_1 \\ g_1e_2 - \sqrt{3}g_2\tilde{e}_2 & -g_1e_1 - g_2\tilde{e}_1 & \sqrt{6}g_2\tilde{e}_3 \end{pmatrix}, \quad (10)$$

where v_d is the vacuum expectation value of H_d. The couplings g_1 and g_2 are complex in general, so we can write $g_2/g_1 = ge^{i\delta}$, where δ is the relative phase of g_1 and g_2. It is convenient to work with the Hermitian matrix $M_e = m_e^\dagger m_e$, to deal only with the left-handed mixing. The matrix M_e can be diagonalized as

$$M_e^{\text{diag}} = U_e^\dagger M_e U_e. \quad (11)$$

At the fixed point $\tau_1 = e^{\frac{2\pi i}{3}}$, the matrix M_e is invariant under ST transformation:

$$(ST)^\dagger M_e ST = M_e, \quad (12)$$

where

$$ST = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -\sqrt{2}e^{\frac{2i\pi}{5}} & \sqrt{2}e^{\frac{-2i\pi}{5}} \\ -\sqrt{2} & -\phi e^{\frac{2i\pi}{5}} & \frac{e^{\frac{-2i\pi}{5}}}{\phi} \\ -\sqrt{2} & \frac{e^{\frac{2i\pi}{5}}}{\phi} & -\phi e^{\frac{-2i\pi}{5}} \end{pmatrix}. \quad (13)$$

Therefore, at the fixed point $\tau_1 = e^{\frac{2\pi i}{3}}$, the A_5 modular group is broken to a $Z_3 = \{1, ST, (ST)^2\}$ residual group. If $g \sim \mathcal{O}(1)$ GeV, $\delta = 0$, the charged lepton mass ratios are $\frac{m_e}{m_\tau} = 0.0003$, $\frac{m_\mu}{m_\tau} = 0.14$, which is consistent with the experimental results.

3.2. Neutrino Sector

For the neutrino sector, we can consider more than one scenario to produce neutrino mass.

(i) Model 1

In this model, the neutrino masses are obtained via the non-renormalizable 5-dimension operator. The neutrino modular A_5 invariant superpotential can be written as

$$w_\nu = \frac{h}{\Lambda} \left(L\, L\, H_u\, H_u\right)_5 Y_5 \left(\tau_\nu\right). \quad (14)$$

After spontaneous symmetry breaking, the scalar fields H_u acquire vev, namely, v_u. The neutrino mass in this case is

$$m_\nu = hv_u^2/\Lambda \begin{pmatrix} 2\tilde{e}_1(\tau) & -\sqrt{3}\tilde{e}_5(\tau) & -\sqrt{3}\tilde{e}_2(\tau) \\ -\sqrt{3}\tilde{e}_5(\tau) & \sqrt{6}\tilde{e}_4(\tau) & -\tilde{e}_1(\tau) \\ -\sqrt{3}\tilde{e}_2(\tau) & -\tilde{e}_1(\tau) & \sqrt{6}\tilde{e}_3(\tau) \end{pmatrix}. \quad (15)$$

The overall factor hv_u^2/Λ can be chosen to get the physical neutrino masses. The matrix m_ν is symmetric but not Hermitian, so it can be diagonalized by two unitary matrices. To deal with the left-handed rotation, one can use the Hermitian matrix

$$M_\nu = m_\nu^\dagger\, m_\nu. \quad (16)$$

This matrix can be diagonalized by one unitary matrix:

$$m_\nu^{\text{diag}} = U_\nu^\dagger M_\nu U_\nu. \quad (17)$$

At the fixed point $\tau_2 = i$, the matrix M_ν is invariant under S transformation:

$$(S)^\dagger M_\nu S = M_\nu. \quad (18)$$

Consequently, A_5 modular group is broken into a $Z_2 = \{1, S\}$ residual group. In this case, the largest two eigenvalues are degenerate and the lightest mass eigenvalue vanishes. So $\tau_2 = i$ does not lead to a physical model. Therefore, deviation from this fixed point should be made to break mass degeneracy. The lepton mixing U_{PMNS} matrix is given by

$$U_{\text{PMNS}} = U_e^\dagger U_\nu. \quad (19)$$

The mixing angles can be calculated from the relations

$$\begin{aligned} \text{Sin}^2\left(\theta_{13}\right) &= \left|\left(U_{\text{PMNS}}\right)_{13}\right|^2, \\ \text{Sin}^2\left(\theta_{12}\right) &= \frac{\left|\left(U_{\text{PMNS}}\right)_{12}\right|^2}{1 - \left|\left(U_{\text{PMNS}}\right)_{13}\right|^2}, \\ \text{Sin}^2\left(\theta_{23}\right) &= \frac{\left|\left(U_{\text{PMNS}}\right)_{23}\right|^2}{1 - \left|\left(U_{\text{PMNS}}\right)_{13}\right|^2}. \end{aligned} \quad (20)$$

For inverted neutrino mass hierarchy, at $\tau_e = e^{2\pi i/3}$, $\tau_\nu = 0.085 + 0.978i$, $g \sim \mathcal{O}(1)$, $hv_u^2/\Lambda \sim \mathcal{O}(10^{-2})$ eV, $\delta = 0$,

$$\begin{gathered} \Delta m_{12}^2 = 7.6 \times 10^{-5}\, \text{eV}^2, \quad \Delta m_{23}^2 = 2.1 \times 10^{-3}\, \text{eV}^2, \\ \frac{m_e}{m_\tau} = 0.0003, \quad \frac{m_\mu}{m_\tau} = 0.14, \\ \theta_{12} = 30.4^\circ, \quad \theta_{23} = 44.26^\circ, \quad \theta_{13} = 8.1^\circ. \end{gathered} \quad (21)$$

(ii) Model 2

Here, we consider three right-handed neutrinos N_i^c, $i = 1, 2, 3$ that transform as in Table 1. In this case, we consider the modular weights as shown in Table 1 to satisfy the following conditions:

$$\begin{aligned} k_L + k_{H_d} + k_E &= 2, \\ k_L + k_{H_u} + k_N &= 2. \end{aligned} \quad (22)$$

The neutrino invariant superpotential under modular A_5 group is

$$w_\nu = h_1 \left(N^c H_u L\right)_3 Y_3 + h_2 \left(N^c H_u L\right)_5 Y_5 + M_R N^c\, N^c. \quad (23)$$

Therefore, the neutrino mass matrices are

$$\begin{aligned} m_D &= v_u \begin{pmatrix} 2h_2\tilde{e}_1 & h_1e_3 - \sqrt{3}h_2\tilde{e}_5 & -h_1e_2 - \sqrt{3}h_2\tilde{e}_2 \\ -h_1e_3 - \sqrt{3}h_2\tilde{e}_5 & \sqrt{6}h_2\tilde{e}_4 & h_1e_1 - h_2\tilde{e}_1 \\ h_1e_2 - \sqrt{3}h_2\tilde{e}_2 & -h_1e_1 - h_2\tilde{e}_1 & \sqrt{6}h_2\tilde{e}_3 \end{pmatrix}, \\ M_R &= f \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \quad (24)$$

The couplings h_1 and h_2 are complex in general, so we can write $h_2/h_1 = h' e^{i\sigma}$, where δ is the relative phase of g_1 and g_2.

The neutrino mass matrix in the basis (ν_L, N^c) is given by

$$M = \begin{pmatrix} 0 & m_D \\ m_D^T & M_R \end{pmatrix}. \tag{25}$$

The light neutrino can be obtained by diagonalizing the above matrix as

$$m_\nu = -m_D {M_R}^{-1} m_D^T. \tag{26}$$

For inverted mass order, with a certain permutation of the eigenvalues of charged lepton mass matrix, at $\tau_e = \tau_1 = e^{2\pi i/3}$, $\tau_\nu = 0.51 + 0.854i$, $g \sim 1.0005$, $h' \sim 2$, $\delta = \pi$, $\sigma = 0$,

$$\begin{gathered} \Delta m_{12}^2 = 7.1 \times 10^{-5}\,\mathrm{eV}^2, \quad \Delta m_{23}^2 = 2.3 \times 10^{-3}\,\mathrm{eV}^2, \\ \frac{m_3}{m_2} = 0.0003, \quad \frac{m_1}{m_2} = 0.14, \\ \theta_{12} = 29.8^\circ, \quad \theta_{23} = 48.3^\circ, \quad \theta_{13} = 8.4^\circ. \end{gathered} \tag{27}$$

(iii) Model 3

In this model, the modular weight of lepton doublet $k_L = 0$, while that of right-handed neutrinos $k_N = 1$. The neutrino invariant superpotential under the modular A_5 group is

$$w_\nu = f\,(N^c H_u L)_1 + m_R\,(N^c\,N^c)_5\,Y_5. \tag{28}$$

So, we end with the following mass matrices:

$$\begin{aligned} m_D &= f \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ m_R &= \begin{pmatrix} 2\tilde{e}_1(\tau) & -\sqrt{3}\tilde{e}_5(\tau) & -\sqrt{3}\tilde{e}_2(\tau) \\ -\sqrt{3}\tilde{e}_5(\tau) & \sqrt{6}\tilde{e}_4(\tau) & -\tilde{e}_1(\tau) \\ -\sqrt{3}\tilde{e}_2(\tau) & -\tilde{e}_1(\tau) & \sqrt{6}\tilde{e}_3(\tau) \end{pmatrix}. \end{aligned} \tag{29}$$

For inverted mass order, at $\tau_e = \tau_1 = e^{2\pi i/3}$, $\tau_\nu = 0.51 + 0.504i$, $g \sim 1.0005$, $\delta = 0$,

$$\begin{gathered} \Delta m_{12}^2 = 7.6 \times 10^{-5}\,\mathrm{eV}^2, \quad \Delta m_{23}^2 = 2.465 \times 10^{-3}\,\mathrm{eV}^2, \\ \frac{m_1}{m_3} = 0.0003, \quad \frac{m_2}{m_3} = 0.14, \\ \theta_{12} = 31.5^\circ, \quad \theta_{23} = 48.3^\circ, \quad \theta_{13} = 8.4^\circ. \end{gathered} \tag{30}$$

4. CONCLUSION

We consider the modular group A_5 to account for lepton masses and mixings. Different assignments of flavors under modular weights lead to different models.

CONFLICTS OF INTEREST

The author declares that there are no conflicts of interest regarding the publication of this paper.

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